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Haar wavelet method for solving FitzHugh-Nagumo equation

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Abstract—In this paper, we develop an accurate and ef£cient Haar wavelet method for well-known FitzHugh-Nagumo equation. The proposed scheme can be used to a wide class of nonlinear reaction-diffusion equations. The power of this manageable method is con£rmed. Moreover the use of Haar wavelets is found to be accurate, simple, fast, ¤exible, convenient, small computation costs and computationally attractive.

Keywords—FitzHugh-Nagumo equation; Haar wavelet method; Adomain decomposition method; Computationally attractive.

I. Introduction

The nonlinear equation proposed by Hodgkin and Huxley [13] is the most widely accepted mathematical description of the excitation and propagation of nerve impulses [3,9,21,28], that is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u-\alpha)(1-u) \tag{1}$$

where is an arbitrary constant. This generally called FitzHugh-Nagumo (FN) equation. The FN system of equations has been derived by both FitzHugh [8] and Nagumo et al. [22]. It is an important nonlinear reaction-diffusion equation used in circuit theory, biology and the area of population genetics [4] as mathematical models. The FN equation describes the dynamical behavior near the bifurcation point for the Rayleigh-Benard convection of binary ¤uid mixtures [24]. When , the FN equation reduces to the real Newell-Whitehead equation. By using Hirota method, Kawahara and Tanaka [16] have found new exact solutions of Eq. (1); by applying the non-classical symmetry reductions approach, Nucci and Clarkson [23] have obtained some new exact solutions with Jacobbi elliptic function. Some other solutions of Eq. (1) have been given by several authors [6,25,27]. Huaying Li and Yucui Guo [20] have obtained the new exact solutions of the FN equation by using £rst integral method. More recently, Abbasbandy [1] proposed the Soliton solutions for the FN equation by using homotopy analysis method. Abdusalam [2] studied the Analytic and approximate solutions for Nagumo telegraph reaction diffusion equation. Angela Slavova and Pietro Zecca [26] have introduced a cellular neural network (CNN) model of FN equation. In this paper, we develop Haar wavelet method for solving Eq. (1). In solving ordinary differential equations by using Haar wavelet related method, Chen and Hsiao [7] had derived an operational matrix of integration based on Haar wavelet. Lepik [17,18,19] had solved higher order as well as nonlinear ODEs and some nonlinear evolution equations by Haar wavelet method. Hariharan et al. [11] have introduced a Haar wavelet method for solving Fisher's equation. We introduce a Haar wavelet method for solving the FitzHugh-Nagumo (FN) equation with the initial and boundary conditions, which will exhibit several advantageous features: i) Very high accuracy fast transformation and possibility of implementation of fast algorithms compared with other known methods. ii) The simplicity and small computation costs, resulting from the sparsity of the transform matrices and the small number of signi£cant wavelet coef£cients. iii) The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically. Beginning from 1980's, wavelets have been used for solution of partial differential equations (PDE). The good features of this approach are possibility to detect singularities, irregular structure and transient phenomena exhibited by the analyzed equations. Most of the wavelet algorithms can handle exactly periodic boundary conditions. The wavelet algorithms for solving PDE are based on the Galerkin techniques or on the collocation method. Evidently all attempts to simplify the wavelet solutions for PDE are welcome. One possibility for this is to make use of the Haar wavelet family. Haar wavelets (which are Daubechies of order 1) consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. A drawback of the Haar wavelets is their discontinuity. Since the derivatives do not exist in the breaking points it is not possible to apply the Haar wavelets for solving PDE directly. There are two possibilities for getting out of this situation. One way is to regularize the Haar wavelets with interpolating splines (e.g. B-splines or Deslaurier-Dabuc interpolating wavelets). This approach has been applied by Cattani [5], but the regularization process considerably complicates the solution and the main advantage of the Haar wavelets-the simplicity gets to some extent lost. The other way is to make use of the integral method, which was proposed by Chen and Hsiao [7]. There are discussions by other researchers [12,14]. The paper is organized the following way. For completeness sake the Haar wavelet method is presented in Section 2. Function approximation is presented in Section 3. The method of solution of the FitzHugh-Nagumo (FN) equation is proposed in Section 4. Some numerical examples are presented in Section 5. Concluding remarks are given in Section 6.

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II. Haar wavelet preliminaries

Haar wavelet is the simplest wavelet. Haar transform or Haar wavelet transform has been used as an earliest example for orthonormal wavelet transform with compact support. The Haar wavelet transform is the £rst known wavelet and was proposed in 1910 by Alfred Haar. They are step functions (piecewise constant functions) on the real line that can take only three values. Haar wavelets, like the well-known Walsh functions (Rao 1983), form an orthogonal and complete set of functions representing discretized functions and piecewise constant functions. A function is said to be piecewise constant if it is locally constant in connected regions.

The Haar transform is one of the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the mean time, several de£nitions of the Haar functions and various generalizations have been published and used. They were intended to adopt this concept to some practical applications as well as to extend its in applications to different classes of signals. Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design.

After discretizing the differential equations in a conventional way like the £nite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

The previous work in system analysis via Haar wavelets was led by Chen and Hsiao [6], who £rst derived a Haar operational matrix for the integrals of the Haar function vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [10], who £rst proposed a Haar product matrix and a coef£cient matrix. Hsiao and Wang proposed a key idea to transform the time-varying function and its product with states into a Haar product matrix. The orthogonal set of Haar function is shown in Fig.1. This is a group of square waves with magnitudes of in certain intervals and zeros elsewhere. For applications of the Haar transform in logic design, ef£cient ways of calculating the Haar spectrum from reduced forms of Boolean functions are needed.

The Haar wavelet family for is de£ned as follows.

$$h_i\left(t\right) = \begin{cases} 1 & for\ t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1 & for\ t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0 & ,elsewhere \end{cases} \quad (2)$$

Integer $m = 2^j \left(j = 1, 2, \ldots J\right)$ indicates the level of the wavelet; k = 0,1,2, $\cdots$, m-1 is the translation parameter. Maximal level of resolution is J. The index i is calculated according to the formula i = m + k + 1 ; in the case of minimal values m=1,k=0, we have i=2, the maximal value of i is $2^m = 2^{(J+1)}$. It is assumed that the value i=1 corresponds to the scaling function for which h_1 . Let us de£ne the collocation points and discretise the Haar function in this way we get the coef£cient matrix , which has the dimension . The operational matrix of integration P, which is a 2M square matrix, is de£ned by the equation

$$\left(PH\right)_{il} = \int_0^{t_l} h_i\left(t\right) dt \quad (3)$$

Any function *y(t)* which is square integrable in the interval [0 ,1) can be expanded in a Haar series with an in£nite number of terms $y(t) = \sum_{i=0}^{\infty} c_i h_i(t),\ i = 2^j + k,\ j \geq 0, 0 \leq k < 2^j,\ t \in [0, 1)$

where the Haar coef£cients

$c_i = 2^j \int_0^1 y(t) h_i(t) \mathrm{d}t$

are determined in such a way that the integral square error

$E = \int_0^1 \left[y(t) - \sum_{i=0}^{m-1} c_i h_i(t)\right]^2 \mathrm{d}t$

is minimum where $m = 2^j$,$j \in \{0\} \cup N$

In general ,for the function *y(t)* to be smooth the series expansion in equation (3) contains an in£nite number of terms. If *y(t)* is a piecewise constant or may be approximated as piecewise constants , then the sum in equation (4) will be terminated after m terms , that is

$$y(t) \cong \sum_{i=0}^{m-1} c_i h_i(t) = c_m^T h_m(t) \quad (4)$$

$$where \quad t \in [0, 1) \;\; and \; c_m \,\underline{\Delta}\, [c_o, c_1, c_2, ..., c_{m-1}]^T \quad (5)$$

where T stands for transposition ,m stands for their dimension.

The £rst four Haar functions can be expressed as follows :

$h_0(t) = [\;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\;]$
$h_1(t) = [\;\; 1 \;\; 1 \;\; -1 \;\; -1 \;\;]$
$h_2(t) = [\;\; 1 \;\; -1 \;\; 0 \;\; 0 \;\;]$
$h_3(t) = [\;\; 0 \;\; 0 \;\; 1 \;\; -1 \;\;]$

If $y(t) = [\;\; 7 \;\; 1 \;\; 3 \;\; 0 \;\;]$ is piecewise constant then ,
$y(t) = \frac{11}{4} h_0(t) + \frac{5}{4} h_1(t) + 3h_2(t) + \frac{3}{2} h_3(t) = c^T H(t)$ where

$$H(t) = H_4(t)\underline{\Delta} \begin{bmatrix} h_0(t) \\ h_1(t) \\ h_2(t) \\ h_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad (6)$$

The Haar coef£cients c_i can be obtained by using (5)directly. c_i can also be obtained by matrix inversion. For this piecewise constant y(t),

$c^T = y(t) H_4^{-1} = \left[\frac{11}{4} \;\; \frac{5}{4} \;\; 3 \;\; \frac{3}{2}\right]$

Equation (12) is called forward transform that is used to obtain wavelet coef£cients.The function y(t) can be recovered from the corresponding wavelet coef£cients and the wavelets $h_i(t)$. Hence (10) is known as inverse transform. As H and H^{-1} contain many zeros , this phenomenon makes the Haar transform much faster than the Fourier and Walsh transforms.

A. *Integration of Haar wavelets :*

In wavelet analysis for a dynamical system, all functions need to be transformed into Haar series. As impulse functions are not preferred (since they are the derivatives of Haar wavelets), Integration of Haar wavelets are preferred, which is expanded into Haar series with coef£cient matrix P [15].

$\int_0^1 h_m(t)dt \cong P_{m\times m}h_m(t),\ t\in[0,1)$ Where m x m square matrix P is called the operational matrix of integration which satis£es the following recursive formula.

$$P_{1\times 1}=\left\langle \frac{1}{2}\right\rangle P_{m\times m}=\frac{1}{2m}\left[\begin{array}{cc} 2mP_{\frac{m}{2}\times\frac{m}{2}} & -H_{\frac{m}{2}\times\frac{m}{2}} \\ H^{-1}_{\frac{m}{2}\times\frac{m}{2}} & O_{\frac{m}{2}\times\frac{m}{2}} \end{array}\right] \tag{7}$$

where $O_{\frac{m}{2}\times\frac{m}{2}}$ is a null matrix of order $\frac{m}{2}\times\frac{m}{2}$

$H_{m\times m}\ \underline{\Delta}\ [h_m(t_0) \quad h_m(t_1)\cdots\ \cdots h_m(t_{m-1})]$

and

$\frac{i}{m}\le t< i+\frac{1}{m}$ and $H^{-1}_{m\times m}=\frac{1}{m}\ H^T_{m\times m}\ diag(r)$

$$r\ \underline{\Delta}\left[1\ 1\ 2\ 2\ 4\ 4\ 4\ 4\ \cdots\ \cdots\underbrace{\frac{m}{2}\ \frac{m}{2}\ \frac{m}{2}\ \cdots\ \cdots\frac{m}{2}}_{\frac{m}{2}\ elements}\right] \tag{8}$$

for $m>2$, proof of equation (14) is found in [15].

III. Function approximation

Any function $y(x)\in L^2(R)$ can be decomposed as

$$y(x)=\sum c_n h_n(x) \tag{9}$$

where the coef£cients c_n are determined by

$$c_n=2^j\int y(x)h_n(x)dx \tag{10}$$

where $n=2^j+k, j\ge 0, 0\le k<2^j$. Specially

$$c_0=\int y(x)dx \tag{11}$$

The series expansion of $y(x)$ contains an in£nite terms. If $y(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then will be terminated at £nite terms, that is

$$y(x)=\sum c_n h_n(x)=c_m^T h_m(x) \tag{12}$$

Where the coef£cients $c^T_(m)$ and the Haar function vector $h_m(x)$ are de£ned as

$c_m^T=[c_o,c_1,...,c_{m-1}]$

and $h_m(x)=[h_0(x),h_1(x),...,h_{m-1}(x)]$

where 'T' means transpose and $m=2^j$.

IV. Method of solution of FitzHugh-Nagumo equation

Consider the FitzHugh-Nagumo equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u(u-\alpha)(1-u) \tag{13}$$

with the initial condition $u(x,0)=f(x), x\in[0,1]$ and the boundary conditions $u(0,t)=g_0(t), u(1,t)=g_1(t), 0<t\le T$

Let us divide the interval (0,1] into N equal parts of length $\Delta t=(0,1]/N$ and denote $t_s=(s-1)\Delta t, s=1,2,...,N$. We assume that $\dot{u}''(x,t)$ can be expanded interms of Haar wavelets as formula

$$\dot{u}''(x,t)=\sum c_s(n)h_n(x)=c^T_{(m)}h_{(m)}(x) \tag{14}$$

where . and ' means differentiation with respect to t and x respectively, the row vector $c^T_{(m)}$ is constant in the subinterval $t\in(t_s,t_{s+1}]$

Integrating formula (14) with respect to t from t_s to t and twice with respect to x from 0 to x , we obtain

$$\dot{u}(x,t)=(t-t_s)c^T_{(m)}h_{(m)}(x)+u''(x,t_s) \tag{15}$$

$$u(x,t)=(t-t_s)c^T_{(m)}Q_{(m)}(x)h_{(m)}(x)$$

$$+u(x,t_s)-u(0,t_s)+x[u'(0,t)-u'(0,t_s)]+u(0,t) \tag{16}$$

$$\dot{u}(x,t)=c^T_{(m)}Q_{(m)}(x)h_{(m)}(x)+x\dot{u}'(0,t)+\dot{u}(0,t) \tag{17}$$

By the boundary conditions, we obtain

$u(0,t_s)=g_0(t_s), u(1,t_s)=g_1(t_s)$

$\dot{u}(0,t)=g_0'(t), \dot{u}(1,t)=g_1'(t)$

Putting in formulae (16) and (17), we have

$$u'(0,t)-u'(0,t_s)=-(t-t_s)c^T_{(m)}Q_{(m)}(x)h_{(m)}(x)$$

$$+g_1(t)-g_0(t)-g_1(t_s)+g_0(t_s) \tag{18}$$

$$\dot{u}'(0,t)=g_1'(t)-c^T_{(m)}Q_{(m)}(x)h_{(m)}(x)-g_0'(t) \tag{19}$$

Substituting formulae (18) and (19) into formulae (15)-(17), and discretizising the results by assuming $x\to x_l$, $t\to t_{s+1}$ we obtain

$$u''(x_l,t_{s+1})=(t_{s+1}-t_s)c^T_{(m)}h_{(m)}(x_l)+u''(x_l,t_s) \tag{20}$$

$$u(x_l,t_{s+1})=(t_{s+1}-t_s)c^T_{(m)}Q_{(m)}(x)h_{(m)}(x_l)+u(x_l,t_s)-g_0(t_s)\ +g_0(t_{s+1})+x_l[-(t_{s+1}-t_s)c^T_{(m)}P_{(m)}f$$

$$+g_1(t_{s+1})-g_0(t_{s+1})-g_1(t_s)+g_0(t_s)] \tag{21}$$

$$\dot{u}(x_l,t_{s+1})=c^T_{(m)}Q_{(m)}(x)h_{(m)}(x)+g_1(t_{s+1})$$

$$+x_l[-c^T_{(m)}P_{(m)}f+g_1'(t_{s+1})-g_0'(t_{s+1})] \tag{22}$$

Where the vector f is de£ned as

$f=[1,\quad \underbrace{0,...,0}_{(m-1)elements}\quad]^T$

In the following the scheme

$$\dot{u}(x_l,t_{s+1})=u''(x_l,ts+1)+u(x_l,ts+1) \tag{23}$$

$[u(x_l,ts+1)-\alpha][1-u(x_l,ts+1)]$

which leads us from the time layer t_s to t_{s+1} is used.

Substituting equations (20)-(23) into the equation (24), we gain

$c^T_{(m)}Q_{(m)}(x)h_{(m)}(x_l)$

$+x_l[-c^T_{(m)}P_{(m)}f+g_1'(t_{s+1})-g_0'(t_{s+1})]+g_0'(t_{s+1})$

$$=u''(x_l,t_{s+1})+u(x_l,t_{s+1})[u(x_l,t_{s+1})-\alpha][1-u(x_l,t_{s+1})] \tag{24}$$

From formula (24) the wavelet coef£cients can be successively calculated.

V. Test Problem

Consider the following problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u-\alpha)(1-u) \qquad (25)$$

with the initial condition $u(x,0) = \lambda, x \in [0,1]$ and the boundary conditions $u(0,t) = g_0(t), u(1,t) = g_1(t), 0 < t \leq T$

The Haar scheme is given by

$$c^T_{(m)}Q_{(m)}(x)h_{(m)}(x_l) \quad +x_l[-c^T_{(m)}P_{(m)}\lambda \ + \ g_1'(t_{s+1}) \ - g_0'(t_{s+1})] + g_0'(t_{s+1})$$

$$= u''(x_l,t_{s+1}) + u(x_l,t_{s+1})[u(x_l,t_{s+1}) - \alpha][1 - u(x_l,t_{s+1})] \qquad (26)$$

From formula (26) the wavelet coef£cients can be successively calculated.

Using Adomian decomposion method, the exact solution in a closed form is given by

$$u(x,t) = 1/1 + e^{\frac{-\zeta}{\sqrt{2}}} \qquad (27)$$

which is full agreement with the results in [15], where $\zeta = x + ct$ and $c = \sqrt{2}[\frac{1}{2} - \alpha]$

TABLE I
COMPARISON OF EXACT AND HAAR SOLUTIONS FOR FITZHUGH-NAGUMO EQUATION AT $t = 0.48$ AND $m = 32$

x	$Exact solution$	$Haar solution (m = 32)$
0.1	0.52274	0.52246
0.2	0.55205	0.55197
0.3	0.58157	0.58135
0.4	0.61118	0.61099
0.5	0.64073	0.64057
0.6	0.67007	0.67045

Computer simulation was carried out in the cases m=16 and m=32, the computed results were compared with the exact solution, more accurate results can be obtained by using a larger m .

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor x86 Family 6 Model 15 Stepping 13 Genuine Intel 1596 Mhz.

VI. Conclusion

The theoretical elegance of the Haar wavelet approach can be appreciated from the simple mathematical relations and their compact derivations and proofs. It has been well demonstrated that in applying the nice properties of Haar wavelets, the differential equations can be solved conveniently and accurately by using Haar wavelet method systematically. In comparison with existing numerical schemes used to solve the nonlinear parabolic equations, the scheme in this paper is an improvement over other methods in terms of accuracy. It is worth mentioning that Haar solution provides excellent results even for small values of ( ). For larger values of (i.e., , , and), we can obtain the results closer to the real values. The Fitzhugh-Nagumo (FN) equation is a special case of the Burgers-Huxley equation. The Burgers-Huxley equation also has special cases where it reduces to the Burgers equation and to the Newell-Whitehead equation. The main goal of this work is to apply the Haar wavelet method to the well-known FitzHugh-Nagumo (FN) equation that appears in many scienti£c applications. The work also con£rmed the power of the Haar wavelet method in handling nonlinear equations in general. This method can be easily extended to £nd the solution of all other non-linear parabolic equations. Another bene£t of our method is that the scheme presented here, with some modi£cations, seems to be easily extended to solve model equations including more mechanical, physical or biophysical effects, such as nonlinear convection, reaction, linear diffusion and dispersion.

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