On Cross-Ratio in some Moufang-Klingenberg Planes

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Abstract—In this paper we are interested in Moufang-Klingenberg planes $M(A)$ defined over a local alternative ring $A$ of dual numbers. We show that a collineation of $M(A)$ preserve cross-ratio. Also, we obtain some results about harmonic points.

Keywords—Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio, harmonic points.

I. INTRODUCTION

In the Euclidean plane, Desargues established the fundamental fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c.300 B.C) is invariant under projection [4, p. 133]. For this reason, cross-ratio is one of the most important concepts of projective geometry.

In this paper we deal with the class (which we will denote by $M(A)$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring $A := A(ε) = A + Ae$ (an alternative field $A$, $ε \notin A$ and $ε^2 = 0$) introduced by Blunck in [8]. We will show that a collineation of $M(A)$ given in [2] preserves cross-ratio. Moreover, we will obtain some results related to harmonic geometry.

The paper is organized as follows: Section 2 includes some basic definitions and results from the literature. In Section 3 we will give a collineation of $M(A)$ from [2] and we show that this collineation preserves cross-ratio. Finally, we obtain some results on harmonic points.

II. PRELIMINARIES

Let $M = (P, L, ε, ∼)$ consist of an incidence structure $(P, L, ε)$ (points, lines, incidence) and an equivalence relation ‘∼’ (neighbour relation) on $P$ and on $L$, respectively. Then $M$ is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:

(PK1) If $P, Q$ are non-neighbour points, then there is a unique line $PQ$ through $P$ and $Q$.

(PK2) If $g, h$ are non-neighbour lines, then there is a unique point $g \cap h$ on both $g$ and $h$.

(PK3) There is a projective plane $M^* = (P^*, L^*, ε)$ and an incidence structure epimorphism $Ψ : M \to M^*$, such that the conditions $Ψ(P) = Ψ(Q) ⇔ P ∼ Q$, $Ψ(g) = Ψ(h) ⇔ g ∼ h$ hold for all $P, Q ∈ P$, $g, h ∈ L$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of $M$.

A Moufang-Klingenberg plane (MK-plane) is a PK-plane $M$ that generalizes a Moufang plane, and for which $M^*$ is a Moufang plane (for the exact definition see [3]).

An alternative ring (field) $R$ is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2b, (ba)a = ba^2, ∀a, b ∈ R.$$  

An alternative ring $R$ with identity element 1 is called local if the set $I$ of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [12, Theorem 3.1]).

Lemma 2.2: The identities

$$x(yzx) = (xy)xz,$$

$$((yxz)x)z = y(xzx),$$

$$xyz = x(yz)$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [11, p. 160]).

We summarize some basic concepts about the coordinatization of MK-planes from [3].

Let $R$ be a local alternative ring. Then $M(R) = (P, L, ε, ∼)$ is the incidence structure with neighbour relation defined

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as follows:

\[
P = \{(x, y, 1) : x, y \in \mathbb{R}\} \\
\cup \{(1, y, z) : y \in \mathbb{R}, \ z \in \mathbb{I}\} \\
\cup \{(w, 1, z) : w, z \in \mathbb{I}\}, \\
L = \{(m, 1, p) : m, p \in \mathbb{R}\} \\
\cup \{(1, n, p) : p \in \mathbb{R}, \ n \in \mathbb{I}\} \\
\cup \{(q, n, 1) : q, n \in \mathbb{I}\} \\
[m, 1, p] = \{(x, xm + p, 1) : x \in \mathbb{R}\} \\
\cup \{(1, zp + m, z) : z \in \mathbb{I}\} \\
[1, n, p] = \{(yn + p, y, 1) : y \in \mathbb{R}\} \\
\cup \{(zp + n, 1, z) : z \in \mathbb{I}\} \\
[q, n, 1] = \{(1, y, yn + q) : y \in \mathbb{R}\} \\
\cup \{(w, 1, wq + n) : w \in \mathbb{I}\} \\
\]

and

\[
\begin{align*}
P &= (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \Leftrightarrow \\
x_1 - y_1 \in \mathbb{I} \ (i = 1, 2, 3), \forall P, Q \in P \\
g = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = h \Leftrightarrow \\
x_1 - y_1 \in \mathbb{I} \ (i = 1, 2, 3), \forall g, h \in L.
\end{align*}
\]

Now it is time to give the following theorem from [3].

**Theorem 2.1:** \(\mathbf{M(\mathbb{R})}\) is an MK-plane, and each MK-plane is isomorphic to some \(\mathbf{M(\mathbb{R})}\).

Let \(\mathbf{A}\) be an alternative field and \(\varepsilon \notin \mathbf{A}\). Consider \(\mathcal{A} := \mathbf{A(\varepsilon)} = \mathbf{A} + \varepsilon \mathbf{A}\) with componentwise addition and multiplication as follows:

\[(a_1 + a_2 \varepsilon)(b_1 + b_2 \varepsilon) = a_1 b_1 + (a_2 b_1 + a_1 b_2) \varepsilon,
\]

where \(a_i, b_i \in \mathbf{A}\) for \(i = 1, 2\). Then \(\mathcal{A}\) is a local alternative ring with ideal \(\mathbb{I} = \varepsilon \mathcal{A}\) of non-units. The set of formal inverses of the non-units of \(\mathcal{A}\) is denoted as \(\mathbb{I}^{-1}\). Calculations with the elements of \(\mathbb{I}^{-1}\) are defined as follows [7]:

\[
\begin{align*}
(a\varepsilon)^{-1} + t &= (a\varepsilon)^{-1} + t + (a\varepsilon)^{-1} \\
q(a\varepsilon)^{-1} &= (aq^{-1})^{-1} \\
(a\varepsilon)^{-1} q &= (q^{-1}a\varepsilon)^{-1} \\
(a\varepsilon)^{-1} &= a\varepsilon,
\end{align*}
\]

where \((a\varepsilon)^{-1} \in \mathbb{I}^{-1}, t \in \mathcal{A}, q \in \mathcal{A} \setminus \mathbb{I}\). (Other terms are not defined.) For more information about \(\mathcal{A}\) and its relation to MK-planes, the reader is referred to the papers of Blunck [7], [8]. In [8], the centre \(\mathbf{Z(\mathcal{A})}\) is defined to be the (commutative, associative) subring of \(\mathcal{A}\) which is commuting and associating with all elements of \(\mathcal{A}\). It is \(\mathbf{Z(\mathcal{A})} := \mathbf{Z(\varepsilon)} = \mathbf{Z} + \varepsilon \mathbf{Z}\), where \(\mathbf{Z} = \{z \in \mathcal{A} : za = z a, \forall a \in \mathcal{A}\}\) is the centre of \(\mathcal{A}\). If \(\mathbf{A}\) is not associative, then \(\mathbf{A}\) is a Cayley division algebra over its centre \(\mathbf{Z}\).

Throughout this paper we assume \(\text{char} \mathbf{A} \neq 2\) and we restrict ourselves to the MK-planes \(\mathbf{M(\mathcal{A})}\) and

Blunck [8] gives the following algebraic definition of the cross-ratio for the points on the line \(g := [1, 0, 0]\) in \(\mathbf{M(\mathcal{A})}\).

\[
\begin{align*}
(A, B; C, D) &= (a, b, c, d) \\
&= \langle (a - d)^{-1}(b - d) \rangle (b - c)^{-1}(a - c) > \\
(Z, B; C, D) &= (z^{-1}, b, c, d) \\
&= \langle (1 - dz)^{-1}(b - d) \rangle (b - c)^{-1}(1 - cz) > \\
(A, Z; C, D) &= (a, z^{-1}; c, d) \\
&= \langle (a - d)^{-1}(1 - dz) \rangle (1 - cz)^{-1}(a - c) > \\
(A, B; Z, D) &= (a, b, z^{-1}, d) \\
&= \langle (a - d)^{-1}(b - d) \rangle (1 - zb)^{-1}(1 - za) > \\
(A, B; C, Z) &= (a, b; c, z^{-1}) \\
&= \langle (1 - za)^{-1}(1 - zb) \rangle (b - c)^{-1}(a - c) >,
\end{align*}
\]

where \(A = (0, a, 1), B = (0, b, 1), C = (0, c, 1), D = (0, d, 1), Z = (0, 1, z)\) are pairwise non-neighbour points of \(g\) and \(< x > = \{y \mid xy : y \in \mathcal{A}\}\).

In [7, Theorem 2], it is shown that the transformations

\[
\begin{align*}
t_u(x) &= x + w; u \in \mathcal{A} \\
r_u(x) &= xu; u \in \mathcal{A} \setminus \mathbb{I} \\
i(x) &= x^{-1} \\
l_u(x) &= ux = (ir_{u}^{-1}i)(x); u \in \mathcal{A} \setminus \mathbb{I}
\end{align*}
\]

which are defined on the line \(g\) preserve cross-ratios. In [6, Corollary (iii)], it is also shown that the group generated by these transformations, which is denoted by \(\Lambda\), equals to the group of projectivities of a line in \(\mathbf{M(\mathcal{A})}\). The elements preserving cross-ratio of the group \(\Lambda\) defined on \(g\) will act a very important role in the proof of Theorem 3.1.

We give the following result from [1, Theorem 8]. This result states a simple way for calculation of the cross-ratio of the points on any line in \(\mathbf{M(\mathcal{A})}\).

**Theorem 2.2:** Let \(\{O, U, V, E\}\) be the basis of \(\mathbf{M(\mathcal{A})}\) where \(O = (0, 0, 1), U = (1, 0, 0), V = (0, 1, 0), E = (1, 1, 1)\) (see [3, Section 4]). Then, according to types of lines, the cross-ratio of the points on the line \(l\) can be calculated as follows:

- If \(A, B, C, D, Z\) are the pairwise non-neighbour points (a) of the line \(l = [m, 1, k]\), where \(A = (a, am + k, 1), B = (b, bn + k, 1), C = (c, cm + k, 1), D = (d, dm + k, 1)\) are not near to the line \(UV = [0, 0, 1]\) and \(Z = (1, m + zp, z)\) is near to \(UV\);  
(b) of the line \(l = [1, n, p]\), where \(A = (an + p, a, 1), B = (bn + p, b, 1), C = (cn + p, c, 1), D = (dn + p, d, 1)\) are not near to \(V\) and \(Z = (n + zp, z)\) near to \(V\);  
(c) of the line \(l = [q, n, 1]\), where \(A = (1, a, q + an), B = (1, b, q + bn), C = (1, c, q + cn), D = (1, d, q + dn)\) are not near to \(V\) and \(Z = (1, 1, zq + n)\) near to \(V\);
then
\[(A, B; C, D) = (a, b, c, d)\]
\[(Z, B; C, D) = (z^{-1}, b, c, d)\]
\[(A, Z; C, D) = (a, z^{-1}; c, d)\]
\[(A, B; Z, D) = (a, b, z^{-1}, d)\]
\[(A, B; C, Z) = (a, b, c, z^{-1})\].

We can give an important theorem, from [1, Theorem 9], about cross-ratio.

**Theorem 2.3**: In \(\text{M}(A)\), perspectivities preserve cross-ratios.

Now we give a definition in \(\text{M}(A)\), well known from the case of Moufang planes [10]. In \(\text{M}(A)\), any pairwise non-neighbour four points \(A, B, C, D \in I\) are called as harmonic if \((A, B; C, D) = z < -1\) and we let \(h(A, B, C, D)\) represent the statement: \(A, B, C, D\) are harmonic.

**III. ON CROSS-RATIO IN \(\text{M}(A)\).**

In this section we will give a collineation of \(\text{M}(A)\), from [2]. Next, we show that the collineation preserves cross-ratios. Now we are ready to give the following collineation:

\[(x, y, 1) \rightarrow (ys^{-1}, xs, 1)\]
\[(1, y, ze) \rightarrow (1, sy^{-1}s, s(y^{-1}z))\] if \(y \notin I\)
\[(1, y, ze) \rightarrow (s^{-1}ys^{-1}, 1, s^{-1}z)\] if \(y \in I\)
\[(w, 1, ze) \rightarrow (1, s, ss, sz)\]

and
\[[m, 1, k] \rightarrow [sn^{-1}s, 1, -(km^{-1})s]\] if \(m \notin I\)
\[[m, 1, k] \rightarrow [1, s^{-1}ms^{-1}, ks^{-1}]\] if \(m \in I\)
\[[1, n, p] \rightarrow [s, s, 1, ps]\]
\[[qe, ne, 1] \rightarrow [sn, s^{-1}, q, 1]\].

Now we are ready to give the following theorem.

**Theorem 3.1**: The collineation \(I_s\) preserve cross-ratio.

**Proof**: Let \(A, B, C, D\) and \(Z\) be the points with the property given in the statement of Theorem 2.2. Then, it is obvious that
\[(A, B; C, D) = (a, b, c, d)\]
\[(Z, B; C, D) = (z^{-1}, b, c, d)\]
\[(A, Z; C, D) = (a, z^{-1}; c, d)\]
\[(A, B; Z, D) = (a, b, z^{-1}, d)\]
\[(A, B; C, Z) = (a, b, c, z^{-1})\],

where \(z \in I\). In this case we must find the effect of \(\varphi\) to the points of any line where \(\varphi\) is the collineations \(I_s\).

Let \(\varphi = I_s\). If \(l = [m, 1, k]\), then
\[\varphi(X) = \varphi(x, zm + k, 1)\]
\[= ((zm + k)s^{-1}, xs, 1)\]
\[\varphi(Z) = \varphi(1, m + zk, z)\]
\[= (1, s(m + zk)^{-1}s, s((m + zk)^{-1}z))\] for \(m + zk \notin I\)
\[\varphi(Z) = \varphi(1, m + zk, z)\]
\[= (s^{-1}(m + zk)s^{-1}, 1, s^{-1}z)\],

for \(m + zk \in I\)
\[\varphi(l) = [sm^{-1}s, 1, -(km^{-1})s]\] for \(m \notin I\)
\[\varphi(l) = [1, s^{-1}ms^{-1}, ks^{-1}]\] for \(m \in I\).

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of \([sn^{-1}s, 1, -(km^{-1})s]\) is as follows:
\[\varphi(A), \varphi(B), \varphi(C), \varphi(D)\]
\[= ((am + k)s^{-1}, (bm + k)s^{-1})\]
\[= (cm + k)s^{-1}, (dm + k)s^{-1})\]
\[= (1, (m + zk)^{-1}s, (bm + k)s^{-1})\]
\[= (s^{-1}(m + zk)s^{-1}, 1, s^{-1}z)\],

where \(\sigma = r_{m^{-1}l \rightarrow o r z} \in A\). From (b) of Theorem 2.2, the cross-ratio of the points of \([1, s^{-1}ms^{-1}, ks^{-1}]\) is as follows:
\[\varphi(A), \varphi(B), \varphi(C), \varphi(D)\]
\[= (as, bs; cs, ds) = (a, b, c, d)\]
\[\varphi(Z) = \varphi(C), \varphi(D)\]
\[= (s^{-1}s, bs; cs, ds) = (s^{-1}, b, c, d)\],

where \(\sigma = r_{m^{-1}l \rightarrow o r z} \in A\).
and
\[ \varphi(l) = [sn, s^{-1}q, 1]. \]

In this case, from (c) of Theorem 2.2, the cross-ratio of the points of \([sn, s^{-1}q, 1]\) is as follows:
\[
\begin{align*}
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= \left(s^{a-1}s, sb^{-1}s, sc^{-1}s, sd^{-1}s\right) \\
&= \sigma(a, b, c, d), \\
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(szs, sb^{-1}s, sc^{-1}s, sd^{-1}s\right) \\
&= \sigma(z^{-1}, b, c, d),
\end{align*}
\]
where \(\sigma = \iota \circ \mu_{a-1} \circ r_{a^{-1}} \in \Lambda\). Consequently, by considering other all cases we get
\[
\begin{align*}
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a, b, c, d) \\
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}, b, c, d) \\
(\varphi(A), \varphi(Z); \varphi(C), \varphi(D)) &= (a, z^{-1}, c, d) \\
(\varphi(A), \varphi(B); \varphi(Z), \varphi(D)) &= (a, b, z^{-1}, d) \\
(\varphi(A), \varphi(B); \varphi(C), \varphi(Z)) &= (a, b, c, z^{-1})
\end{align*}
\]
for collineation \(\varphi\). Combining the last result and the result of (1), the proof is completed.

Now we are ready to give the other results of the paper. On \(\Lambda\) we give the following theorem, an alternate definition of harmonicity and given for an alternative ring \(\mathbb{A}\) with \(\text{char}\mathbb{A} \neq 2\).

**Theorem 3.2:** Let \(a, b, c, d \in \mathbb{A}\). Then \(h(a, b, c, d)\) if and only if
1) if \(a, b, c, d \in \mathbb{A}\), \(2(ab)^{-1} = (a-c)^{-1} + (a-d)^{-1}\).
2) if \(a = z^{-1}\), \(2(d-c)^{-1} + (c-b)^{-1} = z \in \mathbb{I}\).
3) if \(b = z^{-1}\), \(2(c-d)^{-1} + (d-a)^{-1} = z \in \mathbb{I}\).
4) if \(c = z^{-1}\), \(2(b-d)^{-1} + (d-a)^{-1} = z \in \mathbb{I}\).
5) if \(d = z^{-1}\), \(2(a-b)^{-1} + (c-a)^{-1} = z \in \mathbb{I}\).

**Proof:** 1. From the definition of cross-ratio,
\[
h(a, b, c, d) = \left((a-c)^{-1} (b-d) \right) \left((b-c)^{-1} (a-c)\right) = -1.
\]
By direct computation (with Lemma 2.1),
\[
\begin{align*}
(b-c)^{-1} (1-cz) &= -(b-d)^{-1} (1-dz) \\
(b-c)^{-1} (1-cz) &= -(b-d)^{-1} (1-cz + c-dz) \\
(b-c)^{-1} (1-cz) &= -(b-d)^{-1} (1-cz) \\
-(b-d)^{-1} (c-dz) &= (b-c)^{-1} (b-d)^{-1} (1-cz) \\
-(b-d)^{-1} (c-dz) &= -(b-d)^{-1} ((c - d) z) \\
(b-c)^{-1} (b-d)^{-1} (1-cz) &= -(b-d)^{-1} ((c - d) z) \\
(b-c)^{-1} (b-d)^{-1} (1-cz) &= -(b-d)^{-1} ((c - d) z) \\
(b-c)^{-1} (b-d)^{-1} (1-cz) &= -(b-d)^{-1} ((c - d) z) \\
(b-d) (b-c)^{-1} + 1 &= -(c - d) z \\
(b-c + c - d) (b-c)^{-1} + 1 &= -(c - d) z \\
2 + (c - d) (b-c)^{-1} &= -(c - d) z \\
2 (c-d)^{-1} + (b-c)^{-1} &= -z \\
2 (d-c)^{-1} + (c-b)^{-1} &= z \in \mathbb{I},
\end{align*}
\]
where \(zz = 0\) since \(z \in \mathbb{I}\).
3. The proof is same the proof of 2.
4. From the definition of cross-ratio,
\[
h(a, b, z^{-1}, d) = \left((a-d)^{-1} (b-d)\right) \left((1-zb)^{-1} (1-za)\right) = -1.
\]
By direct computation (Lemma 2.1),
\[
\begin{align*}
(1-zb)^{-1} (1-za) &= -(b-d)^{-1} (da) \\
(1+z) (1-za) &= -(b-d)^{-1} (a-b + b-d) \\
1 + zb - za &= -(b-d)^{-1} (a-b) - 1 \\
2 + z (b-a) &= -(b-a)^{-1} (a-b) \\
2(b-a)^{-1} + z &= -(b-d)^{-1} \\
2(b-a)^{-1} + (d-b)^{-1} &= z \in \mathbb{I},
\end{align*}
\]
where \((1-zb)^{-1} = 1 + zb\) and \(zz = 0\).
5. The proof is same the proof of 4.

Now, we give the following theorem, given as without proof in [10] for \(\mathbb{A}\).

**Theorem 3.3:** On \(\Lambda\), the followings is valid:
1) \(h(0, a, 0^{-1}, \frac{a}{2})\)
2) \(h(a, b, 0^{-1}, \frac{b}{2})\)
3) \(h(a, -a, 0^{-1}, 0)\)
4) \(h(1, -1, a, a^{-1})\)
5) \(h(a^2, 1, a, -a)\)

**Proof:** 1. By the definition of cross-ratio, since
\[
\left(0, a, 0^{-1}, \frac{a}{2}\right) = \left(0 - \frac{a}{2} \left(a - \frac{a}{2}\right) = \frac{-2a}{2} = -1,\right.
\]
then \(h(0, a, 0^{-1}, \frac{a}{2})\).
2. By the definition of cross-ratio, since
\[
\left(a, b, 0^{-1}, \frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^{-1} \left(b - \frac{a+b}{2}\right) = \frac{a+b}{2} = -1,
\]
then $h(a, b, 0^{-1}, a + b^{-1})$.

3. By the definition of cross-ratio, since

$$h(-a, 0^{-1}, 0) = (a - 0)^{-1} (-a - 0) = -1,$$

then $h(-a, 0^{-1}, 0)$.

4. By the definition of cross-ratio, since

$$h(1, -1, a, a^{-1}) = \left((1 - a)^{-1} (a - 1)^{-1}\right)$$
$$= \left((-1)^{-1} (1 - a)^{-1}\right)$$
$$= \left((a^{-1} - 1)^{-1} (1 - a^{-1})^{-1}\right)$$
$$= \left((-1)^{-1} + (a + a^{-1})^{-1}\right)$$
$$= \left((a^{-1} - 1)^{-1} - (a - 1)^{-1}\right)$$
$$= \left((-1)^{-1} (a + a^{-1})^{-1}\right)$$
$$= \left((a^{-1} - 1)^{-1} (a + a^{-1})^{-1}\right)$$

then $h(1, -1, a, a^{-1})$.

5. By the definition of cross-ratio, since

$$h(a^2, 1, a, -a) = \left((a^2 + a)^{-1} (1 + a)\right)$$
$$= \left((a + 1)^{-1} (1 + a)\right)$$
$$= \left((-1)^{-1} + (a - 1)^{-1}\right)$$

then $h(a^2, 1, a, -a)$.

REFERENCES


