Definable Subsets in Covering Approximation Spaces

Xun Ge and Zhaowen Li

Abstract—Covering approximation spaces is a class of important generalization of approximation spaces. For a subset \( X \) of a covering approximation space \((U, C)\), is \( X \) definable or rough? The answer of this question is uncertain, which depends on covering approximation operators endowed on \((U, C)\). Note that there are many various covering approximation operators, which can be endowed on covering approximation spaces. This paper investigates covering approximation spaces endowed ten covering approximation operators respectively, and establishes some relations among definable subsets, inner definable subsets and outer definable subsets in covering approximation spaces, which deepens some results on definable subsets in approximation spaces.

Keywords—Covering approximation space, covering approximation operator, definable subset, inner definable subset, outer definable subset.

I. INTRODUCTION

In order to extract useful information hidden in voluminous data, many methods in addition to classical logic have been proposed. Pawlak rough-set theory, which was proposed by Z. Pawlak in [10], plays an important role in applications of these methods. Here, usefulness of approximation spaces has been demonstrated by many successful applications in pattern recognition and artificial intelligence (see [2], [4], [5], [6], [8], [9], [10], [11], [12], [20], for example).

Definition 1.1 ([12]): Let \( U \), the universe of discourse, be a finite set and \( C \) be a partition on \( U \). For \( X \subseteq U \), Put

\[
\underline{C}(X) = \bigcup\{K : K \in C \wedge K \subseteq X\},
\]

\[
\overline{C}(X) = \bigcup\{K : K \in C \wedge K \cap X \neq \emptyset\};
\]

(1) \((U; C)\) is called a covering approximation space.

(2) \( C : 2^U \rightarrow 2^U \) is called lower approximation operator.

(3) \( \overline{C} : 2^U \rightarrow 2^U \) is called upper approximation operator.

(4) \( X \) is called a definable subset of \((U; C)\) if \( \overline{C}(X) = C(X) \).

(5) \( X \) is called a rough subset of \((U; C)\) if \( \overline{C}(X) \neq C(X) \).

Recently, D. Pei generalized definable subsets of approximation spaces to inner definable subsets and outer definable subsets.

Definition 1.2 ([13]): Let \((U; C)\) be a approximation space with approximation operators \( \underline{C} \) and \( \overline{C} \). A subset \( X \) of \( U \) is called an inner (resp. outer) definable subset of \((U; C)\) if \( \underline{C}(X) = X \) (resp. \( \overline{C}(X) = X \)).

For definable subsets, inner definable subsets and outer definable subsets of approximation spaces, D. Pei gave the following result.

Proposition 1.3 ([13]): Let \((U; C)\) be a approximation space with approximation operators \( \underline{C} \) and \( \overline{C} \), and \( X \subseteq U \). Then the following are equivalent.

(1) \( X \) is a definable subset of \((U; C)\).

(2) \( X \) is an inner definable subset of \((U; C)\).

(3) \( X \) is an outer definable subset of \((U; C)\).

Note that, in the past years, approximation spaces and approximation operators have been extended to covering approximation spaces and covering approximation operators respectively (see [1], [3], [7], [14], [15], [16], [17], [18], [19], [20], [21], for example). It is natural to raise the following question.

Question 1.4: Can “approximation space” and “approximation operators” in Proposition 1.3 be replaced by “covering approximation space” and “covering approximation operators” respectively?

For a covering approximation space \((U; C)\), because there are many various covering approximation operators, which can be endowed on \((U; C)\), the answers of Question 1.4 are uncertain, which depend on covering approximation operators endowed on \((U; C)\). In this paper, we investigate covering approximation spaces endowed ten covering approximation operators respectively, and establishes some relations among definable subsets, inner definable subsets and outer definable subsets, which give some answers of Question 1.4 and deepens some results on definable subsets in approximation spaces.

II. PRELIMINARIES

Definition 2.1 ([21]): Let \( U \), the universe of discourse, be a finite set and \( C \) be a family of nonempty subsets of \( U \).

(1) \( C \) is called a cover of \( U \) if \( \bigcup\{K : K \in C \} = U \).

(2) The pair \((U; C)\) is called a covering approximation space if \( C \) is a cover of \( U \).

Definition 2.2: Let \((U; C)\) be a covering approximation space. For \( x \in U \), put \( Md(x) = \{K : x \in K \in C\} \setminus \{x \in S \in C \setminus S \subseteq K \Rightarrow S = K\} \) and \( \mathcal{N}(x) = \bigcap\{K : x \in K \in C\} \). For each \( n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), \( \mathcal{C}_n \) and \( \mathcal{C}_n \) are defined as follows and are called \( n \)-th lower covering approximation operator and \( n \)-th upper covering approximation operator on \((U; C)\) respectively.

(1) \( \mathcal{C}_1(X) = \{K : K \in C \wedge K \subseteq X\} \);

(2) \( \mathcal{C}_2(X) = \overline{C}(X) \cup \{U \mid Md(x) : x \in X - \mathcal{C}_1(X)\} \);

(3) \( \mathcal{C}_3(X) = \bigcup\{K : K \in C \wedge K \subseteq X\} \);
Proposition 2.7: Consider a covering approximation space $(U; C)$ with covering approximation operators $C_n$ and $C_r$, where $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Definition 2.5: Let $(U; C_n)$ be a covering approximation space, where $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $X \subseteq U$.

1. $X$ is called a definable subset of $(U; C_n)$ if $C_n(x) = X$.
2. $X$ is called an inner definable subset of $(U; C_n)$ if $C_n(x) \subseteq X$.
3. $X$ is called an outer definable subset of $(U; C_n)$ if $X \subseteq C_n(x)$.

The following lemma comes from [14], [15], [19], [20], [21].

Lemma 2.6: Let $(U; C_n)$ be a covering approximation space and $X \subseteq U$, where $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then $C_n(X) \subseteq X \subseteq C_n(X)$.

By Definition 2.5 and Lemma 2.6, we have the following proposition.

Proposition 2.7: Let $(U; C_n)$ be a covering approximation space and $X \subseteq U$, where $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then $X$ is a definable subset of $(U; C_n)$ if and only if $X$ is a both inner definable and outer definable subset of $(U; C_n)$.

III. THE MAIN RESULTS

Theorem 3.1: Let $(U; C_1)$ be a covering approximation space and $X \subseteq U$. Then the following are equivalent.

1. $X$ is a definable subset of $(U; C_1)$.
2. $X$ is an inner definable subset of $(U; C_1)$.
3. $X$ is an outer definable subset of $(U; C_1)$.

Proof. (1) $\implies$ (2): Let $X$ be a definable subset of $(U; C_1)$. By Proposition 2.7, $X$ is an inner definable subset of $(U; C_1)$.

(2) $\implies$ (3): Let $X$ be an inner definable subset of $(U; C_1)$. Then $C_1(x) = X$, and so $X \cap C_1(x) = \emptyset$. It follows that $C_1(x) = C_1(x) \cup \{\{U \cup MD(x) : x \in X \cap C_1(x)\}\} = C_1(x)$. So $X$ is an outer definable subset of $(U; C_1)$.

(3) $\implies$ (1): Let $X$ be an outer definable subset of $(U; C_1)$. Then $C_1(x) = X$. By Proposition 2.7, it suffices to prove that $C_1(x) = X$. Since $C_1(x) \subseteq X$ by Lemma 2.6, we only need to prove that $X \subseteq C_1(x)$. Let $y \in X = C_1(x)$. Let $y \in U \subseteq \{U \cup MD(x) : x \in X \cap C_1(x)\}$. If $y \notin C_1(x)$, then $y \in U \subseteq \{U \cup MD(x) : x \in X \cap C_1(x)\}$, and so there is $z \in X \cap C_1(x)$ and $K \subseteq MD(z) \supseteq C$ such that $y \in K$. Since $C \subseteq MD(z) \subseteq C$ such that $y \in K$. Since $K \subseteq MD(z) \subseteq C$ such that $y \in K$. Since $K \subseteq MD(z) \subseteq C$, this proves that $X \subseteq C_1(x)$.

Theorem 3.2: Let $(U; C_2)$ be a covering approximation space and $X \subseteq U$. Consider the following conditions.

1. $X$ is a definable subset of $(U; C_2)$.
2. $X$ is an inner definable subset of $(U; C_2)$.
3. $X$ is an outer definable subset of $(U; C_2)$.

Then (1) $\implies$ (3) $\implies$ (2) $\iff$ (3).

Proof. (1) $\implies$ (3): Let $X$ be a definable subset of $(U; C_2)$. By Proposition 2.7, $X$ is an outer definable subset of $(U; C_2)$.

(3) $\implies$ (2): Let $X$ be an outer definable subset of $(U; C_2)$. Then $C_2(x) = X$. Let $x \in C_2(x)$, then there is $C \subseteq MD(z) \supseteq C$ such that $y \in K$. Since $K \subseteq MD(z) \subseteq C$, this proves that $X \subseteq C_2(x)$. By Proposition 2.7, $X$ is an outer definable subset of $(U; C_2)$. By Proposition 2.7, $X$ is a definable subset of $(U; C_2)$.

(2) $\iff$ (3): Consider a covering approximation space $(U; C_2)$ and $X \subseteq U$, where $U = \{a, b, c\}$, $C = \{\{a, b\}, \{b, c\}\}$, $X = \{a, b\}$. It is not difficult to check that $C_2(x) = X$ and $C_2(x) = U \neq X$. So $X$ is an inner definable subset of $(U; C_2)$ and is not an outer definable subset of $(U; C_2)$.

Theorem 3.3: Let $(U; C_3)$ be a covering approximation space and $X \subseteq U$. Consider the following conditions.

1. $X$ is a definable subset of $(U; C_3)$.
2. $X$ is an inner definable subset of $(U; C_3)$.
3. $X$ is an outer definable subset of $(U; C_3)$.

Then (1) $\implies$ (3) $\implies$ (2) $\iff$ (3).

Proof. (1) $\implies$ (3): Let $X$ be a definable subset of $(U; C_3)$. By Proposition 2.7, $X$ is an outer definable subset of $(U; C_3)$.

(3) $\implies$ (2): Let $X$ be an outer definable subset of $(U; C_3)$. Then $C_3(x) = X$. Let $x \in C_3(x)$, then there is $C \subseteq MD(z) \supseteq C$ such that $y \in K$. It follows that $x \in K \subseteq C_2(x)$. This proves that $X \subseteq C_2(x)$. Since $C_2(x) \subseteq X$ by Lemma 2.6, $C_2(x) = X$. So $X$ is an inner definable subset of $(U; C_3)$.

(2) $\iff$ (3): Consider a covering approximation space $(U; C_3)$ and $X \subseteq U$, where $U = \{a, b, c\}$, $C = \{\{a, b\}, \{b, c\}\}$, $X = \{a, b\}$. It is not difficult to check that $C_3(x) = X$ and $C_3(x) = U \neq X$. So $X$ is an inner definable subset of $(U; C_3)$ and is not an outer definable subset of $(U; C_3)$.

Theorem 3.4: Let $(U; C_4)$ be a covering approximation space and $X \subseteq U$. Then the following are equivalent.

1. $X$ is a definable subset of $(U; C_4)$.
2. $X$ is an inner definable subset of $(U; C_4)$.
3. $X$ is an outer definable subset of $(U; C_4)$.
Proof. (1) $\iff$ (2): Let $X$ be a definable subset of $(U, C_4)$. By Proposition 2.7, $X$ is an inner definable subset of $(U, C_4)$.

(2) $\implies$ (3): Let $X$ be an inner definable subset of $(U, C_4)$.

Then $C_4(X) = X$, and so $X - C_4(X) = \emptyset$. It follows that $C_4(X) = C_4(X) \cup \left( \bigcup \{K : K \in C \land K \bigcap (X - C_4(X)) \neq \emptyset\} \right) = C_4(X) = X$. So $X$ is an outer definable subset of $(U, C_4)$.

(3) $\implies$ (1): Let $X$ be an outer definable subset of $(U, C_4)$. Then $C_4(X) = X$. By Proposition 2.7, it suffices to prove that $C_4(X) = X$. Since $C_4(X) \subset X$ by Lemma 2.6, we only need to prove that $X \subset C_4(X)$. Assume that $X \not\subset C_4(X)$. Then there is $x \in X - C_4(X)$, i.e., $x \in X = C_4(X) = C_4(X) \cup \left( \bigcup \{K : K \in C \land K \bigcap (X - C_4(X)) \neq \emptyset\} \right) \land x \not\in C_4(X)$. So there is $K \in C$ such that $x \in K$ and $K \bigcap (X - C_4(X)) \neq \emptyset$. So $K \subset C_4(X) = X$. It follows that $x \in K \subset C_4(X)$.

This contradicts that $x \not\in C_4(X)$.

**Theorem 3.5:** Let $(U; C_3)$ be a covering approximation space and $X \subset U$. Consider the following conditions.

(1) $X$ is a definable subset of $(U, C_3)$.

(2) $X$ is an inner definable subset of $(U, C_3)$.

(3) $X$ is an outer definable subset of $(U, C_3)$.

Then (1) $\iff$ (2) $\implies$ (3) $\iff$ (2).

**Proof.** (1) $\iff$ (2): Let $X$ be a definable subset of $(U, C_3)$. By Proposition 2.7, $X$ is an inner definable subset of $(U, C_3)$.

(2) $\implies$ (3): Let $X$ be an inner definable subset of $(U, C_3)$. Then $C_3(X) = X$. Since $X \subset C_3(X)$ by Lemma 2.6, we only need to prove that $C_3(X) \subset X$. Let $y \in C_3(X)$. Then there is $K \in C$ such that $y \in K$ and $K \bigcap X \neq \emptyset$. Pick $z \in K \bigcap X$. Then $z \in X = C_3(X) = \{x \in U : \forall K \in C (x \in K \implies K \subset X)\}$. Since $z \in K$, $K \subset X$, and hence $y \in K \subset X$. This proves that $C_3(X) \subset X$.

(3) $\iff$ (1): Let $X$ be an outer definable subset of $(U, C_3)$. Then $C_3(X) = X$. By Proposition 2.7, it suffices to prove that $C_3(X) = X$. Since $C_3(X) \subset X$ by Lemma 2.6, we only need to prove that $X \subset C_3(X)$. Assume that $X \not\subset C_3(X)$. Then $C_3(X) = X$. By Proposition 2.7, it suffices to prove that $C_3(X) \subset X$. Let $y \in C_3(X)$. Then there is $K \in C$ such that $y \in K$ and $K \bigcap X \neq \emptyset$. Pick $z \in K \bigcap X$. Then $z \in X = C_3(X)$. Since $z \in K$, $K \subset X$, and hence $y \in K \subset X$. This proves that $C_3(X) \subset X$.

**Theorem 3.8:** Let $(U; C_3)$ be a covering approximation space and $X \subset U$. Consider the following conditions.

(1) $X$ is a definable subset of $(U, C_3)$.

(2) $X$ is an inner definable subset of $(U, C_3)$.

(3) $X$ is an outer definable subset of $(U, C_3)$.

Then (1) $\iff$ (2) and (3); (2) $\iff$ (3); (3) $\iff$ (2).

**Proof.** (1) $\iff$ (2) and (3): It holds By Proposition 2.7.

(2) $\iff$ (3): Consider a covering approximation space $(U; C_3)$ and $X \subset U$, where $U = \{a, b, c\}$, $C = \{\{a, b\}, \{b, c\}, \{c, a\}\}$, $X = \{a\}$. It is not difficult to check that $C_3(X) = X$ and $C_3(X) = \emptyset \neq X$. So $X$ is an outer definable subset of $(U; C_3)$ and is not an inner definable subset of $(U; C_3)$.

(3) $\iff$ (2): Consider a covering approximation space $(U; C_3)$ and $X \subset U$, where $U = \{a, b, c\}$, $C = \{\{a, b\}, \{b, c\}, \{c, a\}\}$, $X = \{a\}$. It is not difficult to check that $C_3(X) = X$ and $C_3(X) = \emptyset \neq X$. So $X$ is an outer definable subset of $(U; C_3)$ and is not an inner definable subset of $(U; C_3)$.

**Theorem 3.9:** Let $(U; C_3)$ be a covering approximation space and $X \subset U$. Then the following are equivalent.

(1) $X$ is a definable subset of $(U, C_3)$.

(2) $X$ is an inner definable subset of $(U, C_3)$.

(3) $X$ is an outer definable subset of $(U, C_3)$.

**Proof.** (1) $\implies$ (2): Let $X$ be a definable subset of $(U, C_3)$. By Proposition 2.7, $X$ is an inner definable subset of $(U, C_3)$.

(2) $\implies$ (3): Let $X$ be an inner definable subset of $(U, C_3)$. Then $C_3(X) = X$. Since $X \subset C_3(X)$ by Lemma 2.6, it suffices to prove that $C_3(X) \subset X$. Let $y \in C_3(X)$. Then there is $K \in C$ such that $y \in K$ and $K \bigcap X \neq \emptyset$. Pick $z \in K \bigcap X$. Then $z \in X = C_3(X)$. Since $z \in K$, $K \subset X$, and hence $y \in K \subset X$. This proves that $C_3(X) \subset X$.

(3) $\implies$ (1): Let $X$ be an outer definable subset of $(U, C_3)$. Then $C_3(X) = X$. By Proposition 2.7, it suffices to prove that $C_3(X) = X$. Since $C_3(X) \subset X$ by Lemma 2.6, we only need to prove that $X \subset C_3(X)$. Assume that $X \not\subset C_3(X)$. Then $C_3(X) = X$. By Proposition 2.7, it suffices to prove that $C_3(X) \subset X$. Let $y \in C_3(X)$. Then there is $K \in C$ such that $y \in K$ and $K \bigcap X \neq \emptyset$. Pick $z \in K \bigcap X$. Then $z \in X = C_3(X)$. Since $z \in K$, $K \subset X$, and hence $y \in K \subset X$. This proves that $C_3(X) \subset X$. Consequently, $X \subset C_3(X)$. So there is $v \in N(v)$ such that $y \in N(v) \subset X$. Pick $z \in N(v)$ such that $z \not\in X$. Note that $y \in N(v) \subset X$, so
Theorem 3.10: Let $(U; C_{10})$ be a covering approximation space and $X \subset U$. Consider the following conditions.

(1) $X$ is a definable subset of $(U; C_{10})$.

(2) $X$ is an inner definable subset of $(U; C_{10})$.

(3) $X$ is an outer definable subset of $(U; C_{10})$.

Then (1) $\iff$ both (2) and (3); (2) $\not\implies$ (3); (3) $\not\implies$ (2).

Proof. (2) $\not\implies$ (3): Consider a covering approximation space $(U; C_{10})$ and $X \subset U$, where $U = \{a, b, c\}$, $C = \{\{a, b\}, \{b, c\}\}$, $X = \{a, c\}$. It is not difficult to check that $C_{10}(X) = X$ and $\overline{C_{10}}(X) = U \not\subset X$. So $X$ is an inner definable subset of $(U; C_{10})$ and is not an outer definable subset of $(U; C_{10})$.

(3) $\not\implies$ (2): Consider a covering approximation space $(U; C_{10})$ and $X \subset U$, where $U = \{a, b\}$, $C = \{\{a, b\}, \{b, c\}\}$, $X = \{a, b\}$. It is not difficult to check that $C_{10}(X) = X$ and $\overline{C_{10}}(X) = \{a\} \not\subset X$. So $X$ is an outer definable subset of $(U; C_{10})$ and is not an inner definable subset of $(U; C_{10})$.

IV. POSTSCRIPT

In [15], the following covering approximation operators are given for covering approximation spaces.

Definition 4.1: Let $(U; C)$ be a covering approximation space and $X \subset U$. Put

$D(X) = \{x \in U : \exists u \in N(x) \setminus N \subset X\}$;

$\overline{D}(X) = \{x \in U : \forall u \in N(x) \Rightarrow N \subset X \not\subset \emptyset\}$. $D$ and $\overline{D}$ are called lower covering approximation operator and upper covering approximation operator on $(U; C)$ respectively.

We use $(U; D)$ to denote covering approximation space $(U; C)$ with covering approximation operator $D$ and $\overline{D}$.

Definition 4.2: Let $(U; D)$ be a covering approximation space.

(1) $X$ is called a definable subset of $(U; D)$ if $D(X) = \overline{D}(X)$.

(2) $X$ is called an inner definable subset of $(U; D)$ if $D(X) = X$.

(3) $X$ is called an outer definable subset of $(U; D)$ if $D(X) = \overline{D}(X)$.

It is clear that the following proposition holds.

Proposition 4.3: If $X$ is a both inner definable and outer definable subset of covering approximation space $(U; D)$, then $X$ is a definable subset of $(U; D)$.

However, a definable subset of covering approximation space $(U; D)$ need not to be an inner definable or outer definable subset of $(U; D)$. In fact, we have the following example.

Example 4.4: Let $U = \{a, b\}$, $C = \{\{a\}, \{a, b\}\}$, $X = \{a\}$. It is not difficult to check that $D(X) = \overline{D}(X) = U$. $D(X) \not\subset X$ and $\overline{D}(X) \neq X$. It follows that $X$ is a definable subset of $(U; D)$, but $X$ is neither an inner definable nor outer definable subset of $(U; D)$.

Just as above example, for a covering approximation space $(U; D)$ and $X \subset U$, the following implication does not hold in general.

$X \subset U \Rightarrow \overline{D}(X) \subset X \subset D(X)$.

This make it interesting to investigate definable subsets of $(U; D)$. The following question is still open, which is worthy to be considered in subsequent research.

Question 4.5: Let $(U; D)$ be a covering approximation space. Whether there are some relations among definable subsets, inner definable subsets and outer definable subsets of $(U; D)$?

ACKNOWLEDGMENT

This project is supported by the National Natural Science Foundation of China (No. 10971185 and 10971186).

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