Primary subgroups and \( p \)-nilpotency of finite groups

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Abstract—In this paper, we investigate the influence of \( S \)-semipermutable and weakly \( S \)-supplemented subgroups on the \( p \)-nilpotency of finite groups. Some recent results are generalized.

Keywords—\( S \)-semipermutable, weakly \( S \)-supplemented, \( p \)-nilpotent.

I. INTRODUCTION

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [1]. \( G \) denotes always a group, \([G]\) is the order of \( G \), \( \pi(G) \) denotes the set of all primes dividing \([G]\) and \( G_p \) is a Sylow \( p \)-subgroup of \( G \) for some \( p \in \pi(G) \). Two subgroups \( H \) and \( K \) of \( G \) are said to be permutable if \( HK = KH \). A subgroup \( H \) of \( G \) is said to be \( S \)-permutable (or \( S \)-quasinormal, \( S \)-quasinormal) in \( G \) if \( H \) permutes with every Sylow subgroup of \( G \). This concept was introduced by Kegel in [2]. More recently, Q. Zhang and L. Wang generalized \( s \)-permutable subgroups to \( S \)-semipermutable subgroups. \( H \) is said to be \( S \)-semipermutable in \( G \) if \( HG_p = G_pH \) for any Sylow \( p \)-subgroup \( G_p \) of \( G \) with \( (p, |H|) = 1 \) [3]. L. Wang and Y. Wang [4] showed the following theorem: Let \( G \) be a group and \( P \) a Sylow \( p \)-subgroup of \( G \), where \( p \) is the smallest prime dividing \([G]\). If all maximal subgroups of \( P \) are \( S \)-semipermutable in \( G \), then \( G \) is \( p \)-nilpotent. As another generalization of \( s \)-permutable subgroups, Skiba [5] introduced the following concept: A subgroup \( H \) of \( G \) is called weakly \( S \)-supplemented in \( G \) if there is a subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq HS_G \), where \( HS_G \) is the subgroup of \( H \) generated by all those subgroups of \( H \) which are \( S \)-quasinormal in \( G \). In fact, this concept is also a generalization of \( s \)-supplemented subgroups given in [6], Skiba proposed in [5] two open questions related to weakly \( S \)-semipermutable subgroups. In this paper we are concerned with another problems in this context. There are examples to show that weakly \( S \)-supplemented subgroups are not \( S \)-semipermutable subgroups and in general the converse is also false. The aim of this article is to unify and improve some earlier results using \( S \)-semipermutable and weakly \( S \)-supplemented subgroups.

II. PRELIMINARIES

Lemma 2.1. Suppose that \( H \) is an \( S \)-semipermutable subgroup of a group \( G \) and \( N \) is a normal subgroup of \( G \). Then

1. \( H \) is \( S \)-semipermutable in \( K \) whenever \( H \leq K \leq G \).
2. If \( H \) is \( p \)-group for some prime \( p \in \pi(G) \), then \( HN/N \) is \( S \)-semipermutable in \( G/N \).
3. If \( H \leq O_p(G) \), then \( H \) is \( S \)-permutaable in \( G \).

Proof: (a) is [3, Property 1], (b) is [3, Property 2], and (c) is [3, Lemma 3].

Lemma 2.2. ([5], Lemma 2.10) Let \( H \) be a weakly \( S \)-supplemented subgroup of a group \( G \).

1. If \( H \leq L \leq G \), then \( H \) is weakly \( S \)-supplemented in \( L \).
2. If \( N \unlhd G \) and \( N \leq H \leq G \), then \( HN/N \) is weakly \( S \)-supplemented in \( G/N \).
3. If \( H \) is a \( \pi \)-subgroup and \( N \) is a normal \( \pi \)-subgroup of \( G \), then \( HN/N \) is weakly \( S \)-supplemented in \( G/N \).

Lemma 2.3. ([7], A, 1.2) Let \( U, V, W \) be subgroups of a group \( G \). Then the following statements are equivalent:

1. \( U \cap VW = (U \cap V)(U \cap W) \).
2. \( UV \cap UW = U(V \cap W) \).

Lemma 2.4. ([8], Lemma 2.2) If \( P \) is an \( s \)-permutable \( p \)-subgroup of a group \( G \) for some prime \( p \), then \( N_G(P) \geq O^p(G) \).

Lemma 2.5. ([4], Theorem 3.3) Let \( P \) be a Sylow \( p \)-subgroup of a group \( G \), where \( p \) is the smallest prime dividing \([G]\). If every maximal subgroup of \( P \) is \( S \)-semipermutable in \( G \), then \( G \) is \( p \)-nilpotent.

Lemma 2.6. ([10], Lemma 3.4) Let \( H \) be a normal subgroup of a group \( G \) such that \( G/H \) is \( p \)-nilpotent and let \( P \) be a Sylow \( p \)-subgroup of \( H \), where \( p \) is the smallest prime divisor of \( |G| \). If \( |P| \leq p^2 \) and \( G \) is \( A_4 \)-free, then \( G \) is \( p \)-nilpotent.

Lemma 2.7. ([1], IV, 5.4) Suppose that \( G \) is a group which is not \( p \)-nilpotent but whose proper subgroups are all \( p \)-nilpotent. Then \( G \) is a group which is not nilpotent but whose proper subgroups are all nilpotent.

Lemma 2.8. ([1], III, 5.2) Suppose \( G \) is a group which is not \( p \)-nilpotent but whose proper subgroups are all \( p \)-nilpotent. Then

(a) \( G \) has a normal Sylow \( p \)-subgroup \( P \) for some prime \( p \) and \( G = PQ \), where \( Q \) is a non-normal cyclic \( q \)-subgroup for some prime \( q \neq p \).
(b) \( P/\Phi(P) \) is a minimal normal subgroup of \( G/\Phi(P) \).
(c) If \( P \) is non-abelian and \( p > 2 \), then the exponent of \( P \) is \( p \); if \( P \) is non-abelian and \( p = 2 \), then the exponent of \( P \) is 4.
(d) If \( P \) is abelian, then the exponent of \( P \) is \( p \).
(e) \( Z(G) = \Phi(P) \times \Phi(Q) \).

III. MAIN RESULTS

Theorem 3.1. Let \( p \) be the smallest prime divisor of \([G]\) and \( G_p \) be a Sylow \( p \)-subgroup of a group \( G \). If every
maximal subgroup of $G_p$ is either weakly $S$-supplemented or $S$-semipermutable in $G$, then $G$ is $p$-nilpotent.

**Proof:** Suppose that the theorem is false and let $G$ be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $G$ has a unique minimal normal subgroup $N$ and $G/N$ is $p$-nilpotent. Moreover $\Phi(G) = 1$.

Let $N$ be a minimal normal subgroup of $G$. Consider $G/N$. We will show that $G/N$ satisfies the hypothesis of the theorem. Let $M/N$ be a maximal subgroup of $G_pN/N$. It is easy to see $M = G_1N$ for some maximal subgroup $G_1$ of $G_p$. It follows that $G_1N \cap N = G_pN$ is a Sylow $p$-subgroup of $N$. If $G_1$ is $S$-semipermutable in $G$, then $M/N$ is $S$-semipermutable in $G$ by Lemma 2.1. If $G_1$ is weakly $S$-supplemented in $G$, then there is a subgroup $T$ of $G$ such that $G = G_1T$ and $G_1 \cap N \leq (G_1)_G$. So $G/N = M/N \cdot T/N = G_1^N/N \cdot T/N$. Since $\langle (N : N \cdot T \cap N) : (N : T \cap N) \rangle = 1$, we have

$$(G_1 \cap N)(T \cap N) = N = N \cap G = N \cap G_1T.$$ 

By Lemma 2.3, $(G_1N)(T \cap N) = (G_1T)N$. It follows that $(G_1N \cap N)(T \cap N)/N = (G_1N \cap T)(N \cap N) \leq (G_1)_G$. Hence $M/N$ is weakly $S$-supplemented in $G/N$. Therefore, $G/N$ satisfies the hypothesis of the theorem. The choice of $G$ yields that $G/N$ is $p$-nilpotent. Consequently the uniqueness of $N$ and the fact that $\Phi(G) = 1$ are obvious.

(2) $O_p'(G) = 1$. If $O_p'(G) \neq 1$, then $N \leq O_p'(G)$ by step (1). Since $G/O_p'(G) \cong (G/N)/(O_p'(G)/N)$ is $p$-nilpotent, $G$ is $p$-nilpotent, a contradiction.

(3) $O_p(G) = 1$. If $O_p(G) \neq 1$. Step (1) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, $G$ has a maximal subgroup $M$ such that $G = MN$ and $G/N \cong M$ is $p$-nilpotent. Since $O_p(G) \cap M$ is normalized by $N$ and $M$, $O_p(G)/M$ is normal in $G$. The uniqueness of $N$ yields $N = O_p(G)$. Clearly, $G_p = N(G_p \cap M)$. Furthermore $G_p \cap M < G_p$, thus there exists a maximal subgroup $G_1$ of $G_p$ such that $G_p \cap M \leq G_1$. Hence $G_p = NG_1$. By the hypothesis, $G_1$ is either $S$-semipermutable or weakly $S$-permutable in $G$. If we assume $G_1$ is $S$-semipermutable in $G$, then $G_1M_p$ is a group for $q \neq p$. Hence

$$G_1 < M_p, M_q \cap \pi(q) \neq \emptyset \Rightarrow G_1M$$

is a group. Then $G_1M = M$ or $G$ by maximality of $M$. If $G_1M = G$, then $G_p = G_p \cap G_1 = G_1(G_p \cap M) = G_1$, a contradiction. If $G_1M = M$, then $G_1 \leq M$. Therefore, $P_1 \cap M = N$ is of prime order. Then the $p$-nilpotency of $G/N$ implies the $p$-nilpotency of $G$, a contradiction. Therefore we may assume $G_1$ is weakly $S$-supplemented in $G$. Then there is a subgroup $T$ of $G$ such that $G = G_1T$ and $G_1 \cap T \leq (G_1)_G$. From Lemma 2.4 we have $O_p(G) \leq N_G((G_1)_G)$. Since $(G_1)_G$ is subnormal in $G$, we have

$$G_1 \cap T \leq (G_1)_G \leq O_p(G) = N.$$

Thus $(G_1)_G \leq G_1 \cap N$ and $(G_1)_G \leq ((G_1)_G)^G = ((G_1)_G)^{O_p(G)} = ((G_1)_G)^G \leq (G_1 \cap N)^{O_p(G)} = G_1 \cap N \leq N$. It follows that $((G_1)_G)^{O_p(G)} = 1$ or $((G_1)_G)^{O_p(G)} = G_1 \cap N = N$. If $(G_1)_G = G_1 \cap N = N$, then $N \leq G_1$ and $G_p = NG_1 = G_1$, a contradiction. If $(G_1)_G \leq G_1$, then $G_1 \cap T = 1$ and so $T_p = p$. Hence $T$ is $p$-nilpotent. Let $T'_p$ be the normal $p$-complement of $T$. Since $M$ is $p$-nilpotent, we may suppose $M$ has a normal Hall $p'$-subgroup $M_p$ and $M \leq N_G(M_p) \leq G$. The maximality of $M$ implies that $M = N_G(M_p)$ or $N_G(M_p) = G$. If the latter holds, then $M_p \leq G$, and $M_p$ is actually the normal $p$-complement of $G$, which is contrary to the choice of $G$. Hence we may assume $M = N_G(M_p)$. By applying a deep result of Gross([9], main Theorem) and Feit-Thompson’s theorem, there exists $g \in G$ such that $T'_p = M_p$. Hence $T^{g} \leq N_G(T'_p) = N_G(M_p) = G$. However, $T'_p$ is normalized by $T$, so $g$ can be considered as an element of $G_1$. Thus $G = G_1T^{g} = G_1M_p$ and $G_p = G_1(G_p \cap M) = G_1$, a contradiction.

(4) The final contradiction.

If every maximal subgroup of $G_p$ is $S$-semipermutable in $G$, then $G$ is $p$-nilpotent by Lemma 2.5, a contradiction. Thus there is a maximal subgroup $G_1$ of $G_p$ such that $G_1$ is weakly $S$-supplemented in $G$. Then there exists a subgroup $T$ of $G$ such that $G = G_1T$ and

$$G_1 \cap T \leq (G_1)_G \leq O_p(G) = 1.$$ 

By [11, Theorem 2.2], $G$ is not simple and $G$ has a Hall $p'$-subgroup. Suppose $NG_p < G$, then $NG_p$ satisfies the hypothesis of the theorem. The choice of $G$ yields that $N$ is $p$-nilpotent, a contradiction with steps (2) and (3). Therefore we may assume $G = NG_p$. Then we may suppose that $N$ is a Hall $p'$-subgroup $N_p$. By Frattini’s argument, $G = N_G(N_p) = (G_p \cap N_p)N_pNG_p(N_p) = (G_p \cap N_p)NG_p(N_p)$ and so $G_p = G_p \cap G = G_p \cap (G_p \cap N_p)NG_p(N_p) = (G_p \cap N_p)(G_p \cap NG_p(N_p))$. Since $NG_p(N_p) < G$, it follows that $G_p \cap NG_p(N_p) < G_p$. Consider a maximal subgroup $G_1$ of $G_1$ such that $G_p \cap NG_p(G_1) = G_1$. Then $G_p = G_p \cap NG_p(G_1)$. By the hypothesis, $G_1$ is either $S$-semipermutable or weakly $S$-permutable in $G$. If $G_1$ is $S$-semipermutable in $G$, then $G_1NG_p(N_p) = G_1N_p$ forms a group. Since $G : G_1N_p = p$ and $p$ is the smallest prime divisor of $|G|$, we have $G_1N_p \leq G$. By Frattini’s argument again, $G = G_1N_pNG_p(N_p) = G_1NG_p(N_p) < G$, a contradiction. Now assume that $G_1$ is weakly $S$-permutable in $G$. Then there is a subgroup $T$ of $G$ such that $G = G_1T$ and

$$G_1 \cap T \leq (G_1)_G \leq O_p(G) = 1.$$ 

Since $T'_p = p$, we have $T$ is $p$-nilpotent. Let $T'_p$ be the normal $p$-complement of $T$, then $T'_p$ is a Hall $p'$-subgroup of $G$. A application of the result of Gross ([9], Main Theorem)
and Feit-Thompson’s theorem yields $T_{p'}$ and $N_{p'}$ are conjugate in $G$. Since $T_{p'}$ is normalized by $T$, there exists $g \in G$ such that $T_{p'}^g = N_{p'}$. Hence
\[ G = (G_1T)^0 = G_1T^0 = G_1N_G(T_{p'}) = G_1N_G(N_{p'}) \]
and
\[ G_p = G_{p'} \cap G = G_{p'} \cap G_1N_G(N_{p'}) = G_1(G_{p'} \cap N_G(N_{p'})) \leq G_1, \]
a contradiction.

**Theorem 3.2.** Let $p$ be the smallest prime dividing the order of a group $|G|$ and $G_p$ a Sylow $p$-subgroup of $G$. Suppose that $G$ is $A_4$-free and every 2-maximal subgroup of $G_p$ is either weakly $S$-supplemented or $S$-semipermutable in $G$. Then $G$ is $p$-nilpotent.

**Proof.** Suppose that the theorem is false and let $G$ be a counterexample of minimal order. We will derive a contradiction in several steps.

1. By Lemma 2.6, $|G_p| \geq p^3$ and so every 2-maximal subgroups $G_2$ of $G_p$ is non-identity.
2. $G$ has a unique minimal normal subgroup $N$ such that $G/N$ is $p$-nilpotent, Moreover $Φ(G) = 1$.
3. $O_p(G) = 1$.
4. $O_p(G) = 1$. If $O_p(G) = 1$. Step (3) yields $N \leq O_p(G)$ and $Φ(O_p(G)) \leq Φ(G) = 1$. Therefore, $G$ has a maximal subgroup $M$ such that $G = MN$ and $G/N \cong M$ is $p$-nilpotent. Since $O_p(G) \cap M$ is normalized by $N$ and $M$, hence by $G$, the uniqueness of $N$ yields $N = O_p(G)$. Clearly, $G_p = N(G_p \cap M)$. Furthermore $G_p \cap M < G_p$. If $G_p \cap M$ is a maximal subgroup of $G_p$, then $N$ is a subgroup of order $p$. By applying [7, Lemma 2.8], we obtain that $N \leq Z(G)$. Since $G/N$ is $p$-nilpotent, it follows that $G$ is $p$-nilpotent, a contradiction. Therefore $G_p \cap N$ is contained in a 2-maximal subgroup $G_2$. By the hypothesis, $G_2$ is either $S$-semipermutable or weakly $S$-supplemented in $G$. If we assume $G_2$ is $S$-semipermutable in $G$, then $G_2M_q$ is a group for $q \neq p$. Hence
\[ G_2 < M_p, M_q | q ∈ Π(M), q \neq p > = G_2M \]
is a group. Then $G_2M = M$ or $G$ by maximality of $M$. If $G_2M = M$, then $G_p = G_p \cap G_2M = G_2(G_p \cap M)$, a contradiction. If $G_2M = M$, then $G_2 ≤ M$. Therefore, $P_2 \cap N = 1$. Since $G_p = N_{P_2}$, we have $|N| = p^3$. Then the $p$-nilpotency of $G/N$ implies the $p$-nilpotency of $G$ by Lemma 2.6, a contradiction. Now we suppose $G_2$ is weakly $S$-supplemented in $G$. Then there is a subgroup $T$ of $G$ such that $G = G_2T$ and $G_2 \cap T ≤ (G_2)_G$. From Lemma 2.4 we have $O_p(G) ≤ N_G((G_2)_G)$. Since $O_p(G)$ is subnormal in $G$,
\[ G_2 \cap T ≤ (G_2)_G ≤ O_p(G) = N. \]
Thus, $(G_2)_G ≤ G_1 \cap N$, where $p_1$ is a maximal subgroup of $G_p$, which contains $G_2$. Then
\[ (G_2)_G ≤ ((G_2)_G)^0(G_2)_G = ((G_2)_G)^0(G_2)_G ≤ (G_1 \cap N)^0(G_2)_G = G_1 \cap N \leq N. \]
It follows that $((G_2)_G)^0 = 1$ or $((G_2)_G)^0 = G_1 \cap N = N$. If $((G_2)_G)^0 = G_1 \cap N = N$, then $G_2 ≤ G_1$ and $G_p = NG_1 = G_1$, a contradiction. If $((G_2)_G)^0$ is 1, then $G_2 \cap T = 1$ and so $|T_p| = p^2$. Hence $T$ is $p$-nilpotent by Lemma 2.6. Let $T_p$ be the normal $p$-complement of $T$. Since $M$ is $p$-nilpotent, we may suppose $M$ has a normal Hall $p'$-subgroup $M_p$ and $M ≤ NG(M_p) ≤ G$. The maximality of $M$ implies that $M = NG_2(M_p)$ or $NG(M_p) = G$. If the latter holds, then $M_p ≤ G$, $M_p$ is actually the normal $p$-complement of $G$, which is contrary to the choice of $G$. Hence we must have $M = NG_2(M_p)$. By applying a deep result of Gross ([9], main Theorem) and Feit-Thompson’s theorem, there exists $g \in G$ such that $T^g = M_p$. Hence $T^g ≤ NG_2(T_p^g) = NG_2(M_p) = M$. However, $T^g$ is normalized by $T$, so $g$ can be considered as an element of $G_2$. Thus $G = G_2T^g = G_2M$ and $G_p = G_2(G_p \cap M) = G_1$, a contradiction.

5. The final contradiction.
If $NG_p < G$, then $NG_p$ satisfies the hypothesis of the theorem. The choice of $G$ yields that $N$ is $p$-nilpotent, a contradiction with steps (4) and (5). Therefore we must have $G = NG_p$. Since $G/N$ is a $p$-subgroup, we may assume $G$ has a normal subgroup $M$ such that $[G : M] = p$ and $N ≤ M$. Hence the maximal subgroups of $G_p$ are all $2$-maximal subgroups of $G$ satisfying the hypotheses of the theorem. The choice of $G$ yields that $N$ is $p$-nilpotent, a contradiction with steps (4) and (5). Therefore we must have $G = NG_p$. Since $G/N$ is a $p$-subgroup, we may assume $G$ has a normal subgroup $M$ such that $[G : M] = p$ and $N ≤ M$. Hence the maximal subgroups of $G_p$ are all $2$-maximal subgroups of $G$. By applying a deep result of Gross ([9], main Theorem) and Feit-Thompson’s theorem, there exists $g \in G$ such that $T^g = M_p$. Hence $T^g ≤ NG_2(T_p^g) = NG_2(M_p) = M$. However, $T^g$ is normalized by $T$, so $g$ can be considered as an element of $G_2$. Thus $G = G_2T^g = G_2M$ and $G_p = G_2(G_p \cap M) = G_1$, a contradiction.

**Theorem 3.3.** Suppose $N$ is a normal subgroup of a group $G$ such that $G/N$ is $p$-nilpotent, where $p$ is a fixed prime number. Suppose every subgroup of order $p$ of $N$ is contained in the hypercenter $Z_∞(G)$ of $G$. If $p = 2$, in addition, suppose every cyclic subgroup of order $4$ of $N$ is either weakly $S$-supplemented or weakly $S$-semipermutable, then $G$ is $p$-nilpotent.

**Proof.** Suppose that the theorem is false, and let $G$ be a counterexample of minimal order.

1. The hypotheses are inherited by all proper subgroups, thus $G$ is a group which is not $p$-nilpotent but whose proper subgroups are all $p$-nilpotent.

In fact, $K < G$, since $G/N$ is $p$-nilpotent, $K/N K/N \cong K/N$ is also $p$-nilpotent. The cyclic subgroup of order $p$ of $K \cap N$ is contained in $Z_∞(G) \cong K \leq Z_∞(K)$, the cyclic subgroup of order 4 of $K \cap N$ is either weakly $S$-supplemented or $S$-semipermutable in $G$, then is either weakly $S$-supplemented or $S$-semipermutable in $G$ by Lemmas 2.1 and 2.2. Thus $K, K \cap N$ satisfy the hypotheses of the theorem in any case, so $K$ is $p$-nilpotent, therefore $G$ is a group which is not $p$-nilpotent but whose proper subgroups
are all $p$-nilpotent. By Lemmas 2.7 and 2.8, $G = PQ$, $P \leq G$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(2) $G/P \cap N$ is $p$-nilpotent.
Since $G/P \cong Q$ is nilpotent, $G/N$ is $p$-nilpotent and $G/P \cap N \leq G/P \times G/N$, therefore $G/P \cap N$ is $p$-nilpotent.

(3) $P \leq N$.
If $P \not\leq N$, then $P \cap N < P$. So $Q(P \cap N) < QP = G$. Thus $Q(P \cap N)$ is nilpotent by (1), $Q(P \cap N) = Q \times (P \cap N)$. Since $G/P \cap N = P/P \cap N \cdot Q(P \cap N)/P \cap N$, it follows that $Q(P \cap N)/P \cap N \leq G/P \cap N$ by Step (2). So $Q$ char $Q(P \cap N) \subseteq G$. Therefore, $G = P \times Q$, a contradiction.

(4) $p = 2$.
If $p > 2$, then $\exp(P) = p$ by (a) and Lemma 2.9. Thus $P = P \cap N \leq Z_\infty(G)$. It follows that $G/Z_\infty(G)$ is nilpotent, and so $G$ is nilpotent, a contradiction.

(5) For every $x \in P \setminus \Phi(P)$, we have $o(x) = 4$.
If not, there exists $x \in P \setminus \Phi(P)$ and $o(x) = 2$. Denote $M = \langle x^2 \rangle \leq P$. Then $M/\Phi(P)/\Phi(P) \leq G/\Phi(P)$, we have that $P = M \Phi(P) = M \leq Z_\infty(G)$ as $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ by Lemma 2.9, a contradiction.

(6) For every $x \in P \setminus \Phi(P)$, $\langle x \rangle$ is weakly $S$-supplemented in $G$.
If $\langle x \rangle$ is $S$-semipermutable in $G$, then $\langle x \rangle$ is $S$-permutable in $G$ by Lemma 2.1(4), and so weakly $S$-supplemented in $G$.

(7) Final contradiction.
For any $x \in P \setminus \Phi(P)$, we may assume that $x$ is weakly $S$-supplemented in $G$ by Step (6). Then there is a subgroup $T$ of $G$ and $\langle x \rangle$ such that $G = \langle x \rangle \cap T \leq \langle x \rangle$ and $\langle x \rangle \cap T \leq \langle x \rangle \cap G$. It follows that $P = P \cap G = P \cap \langle x \rangle \cap T = \langle x \rangle \cap (P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)/\Phi(P)/\Phi(P) \subseteq G/\Phi(P)$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P = (P \cap T)/\Phi(P) = P \cap T$. If $P \cap T \leq \Phi(P)$, then $\langle x \rangle = P \leq G$, a contraction. If $P = (P \cap T)/\Phi(P) = P \cap T$, then $T = G$ and so $\langle x \rangle = \langle x \rangle \cap G$ is $S$-permutable in $G$. We have $\langle x \rangle \cap G$ is a proper subgroup of $G$ and so $\langle x \rangle \cap Q = \langle x \rangle \cap \times Q$, i.e., $\langle x \rangle \cap N_G(Q)$. By Lemma 2.8, $\Phi(P) \subseteq Z(G)$. Therefore we have $P \leq N_G(Q)$ and so $Q \leq G$, a contradiction.

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