Bounds On The Second Stage Spectral Radius Of Graphs

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Abstract—Let G be a graph of order n. The second stage adjacency matrix of G is the symmetric \( n \times n \) matrix for which the \( ij^{th} \) entry is 1 if the vertices \( v_i \) and \( v_j \) are of distance two; otherwise 0. The sum of the absolute values of this second stage adjacency matrix is called the second stage energy of G. In this paper we investigate a few properties and determine some upper bounds for the largest eigenvalue.

Keywords—Second stage spectral radius; Irreducible matrix; Derived graph.

I. INTRODUCTION

Let G be a connected graph with vertex set \( V(G) = \{v_1, v_2, ..., v_n\} \). The second stage adjacency matrix is denoted by \( A_2(G) \) and the second stage energy by \( E_2(G) \). As it is symmetrical it will be an adjacency matrix for some graph \( G' \) which we call the derived graph of G. If \( \Delta' \) is the maximum degree of \( G' \) then clearly \( \Delta' \leq \Delta \). Irreducibility of the adjacency matrix is related to the property of connectedness[2]. Hence \( A_2(G) \) is irreducible if and only if the derived graph \( G' \) is connected. Proposition 2.1 guarantees plenty of graphs for which their derived graphs are connected, for example, the Peterson graph whose derived graph is a 6-regular graph. In this paper we consider only those graphs for which \( A_2(G) \) is irreducible.

II. SOME PROPERTIES

The derived graph of any odd cycle \( C_{2m-1} \) is irreducible. Proposition 2.1. Let G be a graph having \( C_{2m-1} \) as an induced subgraph for some \( m \geq 3 \). Then \( G' \) is connected.

Let \( u = u_1u_2...u_r = v_i \) be the shortest path from \( u \) to \( C_{2m-1} \) of length \( r \).

Case 1. r is even, then we have

\[ d(u = u_1u_3...u_r = v_i) = 2 \]

and the derived graph has the path \( u_1u_3u_5...u_{r-2}v_i \).

Case 2. r is odd, then we have \( u_1u_3u_5...u_{r-1}v_i \) is a path in the derived graph.

Remark: Converse of the above proposition is not true. For example, consider any odd cycle other than \( C_5 \). It is 2-regular and its derived graph being an odd cycle is also 2-regular. But \( n \neq 2r + 1 \).

Proposition 2.2. Let G be a r-regular graph with order \( n \) such that \( n = 2r + 1 \). Then the derived graph \( G' \) of G is also r-regular.

Proof: Clearly r is even. Choose any vertex \( v_i \). Let \( v_k \) be a vertex such that \( v_k \notin N(v_i) \).

Claim: \( d(v_k, v_i) = 2 \). Otherwise, \( v_k \notin N(v_i) \) for all \( v_j \in N(v_i) \). This implies \( deg(v_k) \leq 2r + 1 - (r + 2) = r - 1 \), which is a contradiction since \( deg(v_k) = r \).

Proposition 2.3. The derived graph of circulant graph is a circulant graph.

Proof: Let G be a circulant graph formed by the set \( S = \{1, 2, ..., n\} \). Then for each fixed \( i \in S \) and only if \( n - i \in S \) [1].

Consider a vertex \( v_i \). Let \( v_k \in D(v_i) \). Then there exists a vertex \( v_j \) such that \( v_j \) is adjacent to \( v_i \) and \( v_k \). Then by the definition of circulant graph, \( v_{n-k} \) is also adjacent to \( v_j \) and so \( v_{n-k} \in D(v_i) \). Thus, \( G' \) is formed by a set \( S' = \{1, 2, ..., n\} \) such that \( k \in S' \) if and only if \( n - k \in S' \) and hence \( G' \) is also circulant.

Proposition 2.4. Given any positive integer \( n \) of the form \( p^r \) where \( p \) is a prime number and \( r \) is a positive integer, there exists a graph G for which the second stage energy is \( 2(p-1)p^{r-1} \).

Proof: Let G be the complement of the circulant graph H formed by the set \( S = \{\alpha_1, \alpha_2, ..., \alpha_k\} \) where \( \alpha_i \)'s are all numbers less than \( n \) and prime to \( n \). Then the derived graph of G is the circulant graph H whose energy is \( 2(p-1)p^{r-1} \) [1]. Hence \( E_2(G) = E(H) = 2(p-1)p^{r-1} \).

Theorem 2.5. Let \( D(v_i) = \{v_j : d(v_i, v_j) = 2\} \). Then for each fixed \( i = 1, 2, ..., n, \) \( |D(v_i)| = S_1 - S_2 \), where

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Hence

\[ \rho(G)^2 = \sum_{i=1}^{n} \rho(G_j)^2 x_i^2 \]

\[ \leq \sum_{i=1}^{m} (2m - (2d_i + \delta - \epsilon_F)) \sum_{j \in D(v_i)} x_j^2 \]

\[ = \sum_{i=1}^{m} (2m - (2d_i + \delta - \epsilon_F)) - \sum_{i=1}^{n} (2m - (2d_i + \delta - \epsilon_F)) \sum_{j \in D(v_i)} x_j^2 \]

\[ = 2mn - 4m - n\delta + \epsilon_F \sum_{i=1}^{m} (2m - (2d_i + \delta - \epsilon_F)) \sum_{j \in D(v_i)} x_j^2 \]

In (3), we estimate, \(-\sum_{i=1}^{n} (2m - (2d_i + \delta - \epsilon_F)) \sum_{j \in D(v_i)} x_j^2\)

Now, consider

\[ \sum_{i=1}^{m} (2d_i + \delta - \epsilon_F) \sum_{j \in D(v_i)} x_j^2 \]

\[ = \sum_{i=1}^{m} (2d_i + \delta - \epsilon_F) x_i^2 \]

\[ \geq \sum_{i=1}^{m} (2d_i + \delta - \epsilon_F) x_i^2 \]

\[ \sum_{i=1}^{m} (2d_i + \delta - \epsilon_F) x_i^2 \]

\[ \sum_{i=1}^{m} (n - (d_i - \epsilon_F) - 1) x_i^2 \]

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Let $P$ be any polynomial. If both inequalities are strict,

\[\rho(G) \leq \sqrt{2m(n - \delta - 1) - 4m + \delta(2 - \delta) + A}, \text{ where } A = \varepsilon F(2\Delta + 1).\]

Let $B$ be an $n \times n$ matrix and let $S_i(B)$ denote the $i$th row sum of $B$, i.e., $S_i(B) = \sum_{j=1}^{n} B_{ij}$, where $1 \leq i \leq m$.

Lemma 3.2. Let $G$ be a connected $n$-vertex graph and $A_2$ its second stage adjacency matrix, with spectral radius $\rho$.

Proof. Since $A_2$ is irreducible, the proof is just analogous to that of Lemma 2.2 in [4].

Lemma 3.3. For each fixed $i=1,2, \ldots, n$,

\[S_i(A_2^2) = |D(v_i)| + \sum_{j \neq i} \sum_{k=1}^{n} a_{ik} a_{kj} - \{v_k : d(v_k, v_i) = 2 \text{ and } d(v_k, v_j) = 2\} \]

Proof: $ij$'th entry in $b_{ij}$ in $A_2^2 = \sum_{k=1}^{n} a_{ik} a_{kj}$

Case 1. Let $i = j$, then $b_{ii} = \sum_{k=1}^{n} a_{ik} a_{ki} = |D(v_i)|$.

Case 2. Let $i \neq j$, $a_{ik} a_{kj} = 1$ if and only if $a_{ik} = 1$ and $a_{kj} = 1$.

Let $G$ be a simple graph with $n$ vertices and $m$ edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of $G$ and $\rho(G)$ be the spectral radius of the adjacency matrix $A$ of $G$. Then in [6] it is proved that,

\[\rho(G) \leq (\delta - 1 + \sqrt{\delta + 1})^2 + 2(2m - \delta n)/2.\]

Theorem 3.4. Let $G$ be a simple graph with $n$ vertices and $m$ edges. Let $\Delta = \Delta(G)$ be the maximum degree of vertices of $G$ and $\rho(G)$ be the spectral radius of the second stage adjacency matrix $A_2$ of $G$. Then $\rho(G) \leq (1 + \sqrt{4(n - 1)\Delta}/2).$ Proof: Since $S_i(A_2^2) = |D(v_i)| + \sum_{j \neq i} \sum_{k=1}^{n} \{v_k : d(v_k, v_i) = 2 \text{ and } d(v_k, v_j) = 2\} - S_i(A_2) = \sum_{j \neq i} \sum_{k=1}^{n} \{v_k : d(v_k, v_i) = 2 \text{ and } d(v_k, v_j) = 2\} \leq (n - 1)\Delta$. As this holds for every vertex $v \in V(G)$. Lemma 3.2 implies that $\rho(G^2) - \rho(G) \leq (n - 1)\Delta$. Solving the quadratic inequality, we obtain $\rho(G) \leq (1 + \sqrt{4(n - 1)\Delta}/2).

For a non regular graph, many upper bounds for the largest eigenvalue of adjacency matrix are found. One such upper bound is

\[\lambda_1 \leq \Delta - (1/2m(n - \Delta - 1)\Delta^2) [5].\]

In the following theorem we find a similar upper bound for our second stage concept.

Theorem 3.5. If $G$ is connected and not regular, then

\[\lambda_1 \leq \Delta - (1/4\Delta^2 n(2m - 3\delta + \varepsilon F)).\]

Proof: Let $x$ be a positive unit eigenvector of $A_2(G)$ corresponding to $\lambda_1$. We have that $|x|^2 = \lambda_1 \sum_{v \in V} |x_v|^2 = 2 \sum_{v \in V} x_v x_i x_j$.

Since the maximum degree of $G$ is $\Delta$ and $G$ is not regular, we have

\[\Delta = \max_{v \in V} |D(v)| \geq 2m - 2d_v - \delta + \varepsilon F, \text{ it follows that } \sum_{v \in V} x_v x_i^2 \geq (1/|D(v_i)|)(\sum_{v \in V} x_i x_j^2) \geq (1/2m - 2d_v - \delta + \varepsilon F)(\sum_{v \in V} x_i^2 x_j^2) \geq (1/2m - 3\delta + \varepsilon F)(\sum_{v \in V} x_i x_j^2) \geq (1/2m - 3\delta + \varepsilon F)(\sum_{v \in V} x_i x_j^2) \geq (1/2m - 3\delta + \varepsilon F)(\sum_{v \in V} x_i^2 x_j^2) \geq (1/2m - 3\delta + \varepsilon F)(\sum_{v \in V} x_i x_j^2).

Let $u$ and $v$ be the vertices of derived graph $G$ such that $x_u = \max_{v \in V} x_v$ and $x_v = \min_{v \in V} x_v$. Let $u = w_0 w_1 \ldots w_k = v$ be a path between $u$ and $v$ in the derived graph $G$. Then

\[\sum_{v \in V} x_v x_i x_j \geq \sum_{k=0}^{k-1} x_{w_k} x_{w_{k+1}} \geq \sum_{k=0}^{k-1} x_{w_k} x_{w_{k+1}} \geq \sum_{k=0}^{k-1} x_{w_k} - x_{w_{k+1}} \geq x_{w_0} - x_{w_k} = x_u - x_v.

We have $\Delta - \lambda_1 > (1/2m - 3\delta + \varepsilon F)(x_u - x_v)^2$. It remains to estimate $x_u - x_v$. Since $\sum_{v \in V} x_v^2 = 1$, we have $x_u \geq 1/\sqrt{n}$ and $x_v \leq 1/\sqrt{n}$. There are three cases to consider.

Case Ia: $x_u \geq 1/\sqrt{n} + c$. Then $x_v < 1/\sqrt{n}$ and $\Delta - \lambda_1 > (c^2/2m - \delta + \varepsilon F).

Case Ib: $x_v \leq 1/\sqrt{n} - c$. Then $x_u > 1/\sqrt{n}$ and again $\Delta - \lambda_1 > (c^2/2m - \delta + \varepsilon F).

Case Ii: $1/\sqrt{n} - c < x_v < x_u < 1/\sqrt{n} + c$. Then $x_v \in (1/\sqrt{n} - c, 1/\sqrt{n} + c)$. Then $x_v \in (1/\sqrt{n} - c, 1/\sqrt{n} + c)$ holds for each $v \in V$, and by choosing $s \in V'$ with $d_s < \Delta - 1$, which is regular, we get

$\lambda_1(1/\sqrt{n} - c) \leq \lambda_1 < \lambda_s$.
\[
= \sum_{t; (s,t) \in E'} x_t < (\Delta' - 1)(1/\sqrt{n} + c)
\]
where \(\Delta' = \max D(v_i), \ i = 1, 2, ..., n\) and \(E'\) is the edge set of \(G'\) which implies,
\[\lambda_1 < (\Delta' - 1)(1 + c\sqrt{n}/1 - c\sqrt{n}).\]
In order for the expression on the RHS to be useful, it must be less than \(\Delta'\), which is satisfied for \(c < 1/(2\Delta' - 1)\sqrt{n}\). Put \(c = 1/2\Delta'\sqrt{n}\) in cases Ia and Ib, we get,
\[
\Delta - \lambda_1 > \frac{1}{2m - 3\delta + \epsilon_F} \frac{4(\Delta')^2 n}{n}
\]
\[
\lambda_1 < \Delta - \frac{1}{2m - 3\delta + \epsilon_F} \frac{4(\Delta')^2 n}{n}
\]
While in \(\lambda_1 < (\Delta' - 1)(1 + \sqrt{n}/2\Delta'\sqrt{n}/1 - \sqrt{n}/2\Delta'\sqrt{n})\)
\[
< (\Delta' - 1)(2\Delta' + 1/2\Delta' - 1)
= (2\Delta' + \Delta' - 2\Delta' - 1/2\Delta' - 1)
= \Delta' - (1/2\Delta' - 1)
\]
This implies \(\lambda_1 < \Delta' - (1/2\Delta' - 1)\)
\[
< \Delta - \frac{1}{4(\Delta')^2(2m - 3\delta + \epsilon_F)}
\]
\[
< \Delta - \frac{1}{4(\Delta)^2 n(2m - 3\delta + \epsilon_F)}
\]

REFERENCES