Numerical Study of Vertical Wall Jets: Influence of the Prandtl Number

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Abstract—This paper is a numerical investigation of a laminar isothermal plane two dimensional wall jet. Special attention has been paid to the effect of the inlet conditions at the nozzle exit on the hydrodynamic and thermal characteristics of the flow. The behaviour of various fluids evolving in both forced and mixed convection regimes near a vertical plate plane is carried out. The system of governing equations is solved with an implicit finite difference scheme. For numerical stability we use a staggered non uniform grid. The obtained results show that the effect of the Prandtl number is significant in the plume region in which the jet flow is governed by buoyant forces. Further for ascending X values, the buoyancy forces become dominating, and a certain agreement between the temperature profiles are observed, which shows that the velocity profile has no longer influence on the wall temperature evolution in this region. Fluids with low Prandtl number warm up more importantly, because for such fluids the effect of heat diffusion is higher.

Keywords—Forced convection, Mixed convection, Prandtl number, Wall jet.

I. INTRODUCTION

Due to the fact of the diversity of their aspects and their applications, typical jet flows represent a constant interest. Indeed, they are widely used in many industrial applications, such as jet cutting, cooling through impingement, heat insulation, inlet devices in ventilation, separation control in airfoils and film cooling of turbine blades [1].

The wall jet in which the flow is injected tangentially to a plane plate at high velocity, the inner wall boundary and the outer free jet attracted much attention of researchers as indicated by an abundant literature on related subjects. Thus, this type of flow has been the subject of several studies that have been devoted to both hydrodynamic and thermal characteristic of the flow, particularly to know the local coefficient of heat transfer.

In such plane wall jets, we distinguish: the zone of the boundary layer which lies between the wall and the line maximum velocity and the external zone that behaves like a free jet [2,3]. Former works were concerned with this type of flows but the majority treated the turbulent case, due to its practical interest for industrial applications. Nevertheless laminar wall jets are widely used in various industrial and domestic application such as the filling and the cleaning of the surfaces coating by air flow and continuous lubrication of the contact of solid surfaces in order to avoiding wear.

In turbulent mode Nizou [2] and then Nizou and Tida [3] proposed in their experiments an analytical formulation for the parietal friction coefficient as a function of the longitudinal distance of the flow. This allows the deduction of an analogy between the transfer of heat and momentum for a turbulent parietal jet. Leduc et al. [4] proposed numerical solutions relating to the thermal transfer between a jet in forced convection and a plane plate subjected to a constant heat flux. Lauder and Spalding [5] established the system of equations recommended for high Reynolds numbers. Ljuboja and Rodi [6] showed that this model presents limitations for the parietal jet and proposed a modified version of the k-ε model.

Recently Kechiche et al. [7], [8] studied the influence of the velocity and temperature conditions at the nozzle exit on the dynamic, thermal and turbulent parameters of the flow. For the case of an isothermal parietal jet, they showed that these conditions do not have an influence on the flow parameters in the established area of the mode. They showed that in the case of an air blast evolving tangentially with a plate subjected to a constant heat flux in forced convection, the velocity profile at the nozzle exit does not have an influence on the thermal transfer between the plate and the flow in the thermal established region. A reduction in the Reynolds number produces a reduction in the convective thermal transfer.

An examination of the literature reveals that in laminar mode, the large majority of the studies limit itself to analytical solutions or a numerical resolution using a change of variables allowing to ignore the conditions at the nozzle exit. We can quote for example, Schlichting [9] and Gorla [10]; these authors analyzed mainly the established zone of the jet while basing themselves on the momentum conservation and on the fact that, in the mixture zone, the inertia forces are dominating the buoyancy forces.

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Following the work of Savage and Chan [11] who separately studied the area of the jet and that of the plume, Wilks and Hunt [12], by using a more adequate variables change, proposed numerical results for the three regions of a plane wall jet, exact analytical solutions for the forced convection in the vicinity of the nozzle and for the natural convection far from the emission section were deduced.

Yu et al. [13] studied numerically a laminar wall jet by introducing new parameters and a variables change which replace the two initial forms of the velocity and temperature profiles at the nozzle exit. These conditions were replaced by two integration constraints expressing the momentum and energy conservation of the flow discharged by the nozzle.

Later Mhiri et al. [14] were interested in the effect of the emission conditions at the exit of the nozzle on isothermal and non-isothermal laminar wall jet. They considered two emission conditions verifying the constraints suggested by Yu et al. [13]. They showed that in the case of a laminar wall jet these conditions do not have any influence on the flow in the plume zone, where the buoyancy forces are dominating. It is reached at a critical distance depending on the Re and Gr numbers.

A more detailed numerical study of the influence of an initial disturbance on the evolution of dynamic and thermal parameters of a pulsed laminar wall jet was made by Marzouk et al. [15]. They used two wall thermal conditions: adiabatic or isotherm. They showed that for flow pulsation amplitude of 10% and a Strouhal number of 0.3, an increase of 8% in the static pressure effects are negligible; thus the problem can be assumed two-dimensional. The experiment shows that the static pressure on the evolution of dynamic and thermal characteristics of the jet for various fluids evolving in forced convection near a vertical plate plane: two velocity profiles, uniform and parabolic, are tested at the exit of the nozzle.

II. ASSUMPTIONS AND GOVERNING EQUATIONS

We consider an incompressible laminar jet resulting from a rectangular nozzle into a quiescent surrounding fluid. The nozzle thickness is smaller compared to its width so that edge effects are negligible; thus the problem can be assumed two-dimensional. The experiment shows that the static pressure undergoes a very weak variation, so we consider it was constant in the jet. Both mixed and forced convection regimes are considered by using the Boussinesq approximation in which the density varies linearly with temperature in the buoyancy term of the momentum conservation equation. The geometry and the coordinates system used is shown on figure 1.

Under the above assumptions, the governing equations describing the behaviour of a vertical buoyant wall jet flow are:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(1)

These equations are completed by the following set of boundary conditions:

\[
x = 0 \quad 0 < y < w : \quad u = u_0 ; \quad v = 0 ; \quad T = T_0 ;
\]

\[
Y \geq w : \quad u = 0 ; \quad v = 0 ; \quad T = T_0 ;
\]

\[
x > 0 \quad y = 0 : \quad u = 0 ; \quad v = 0 ; \quad \left( \frac{\partial (T - T_0)}{\partial y} \right)_{y=0} = -\frac{\phi}{k} ;
\]

\[
y \to \infty : \quad u = 0 ; \quad T = T_0 ;
\]  

(4)
By using the following dimensionless variables:
\[
X = \frac{x}{w} ; \quad Y = \frac{y}{w} ; \quad U = \frac{u}{u_0} ; \quad V = \frac{v}{u_0} \quad \text{and} \quad \theta = \frac{T - T_c}{\phi w}
\]
These equations become, in their dimensionless form:

\[
\frac{\partial U}{\partial X} + \frac{V}{\partial Y} = 0
\]

(6)

\[
U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \frac{1}{Re} \left( \frac{\partial^2 U}{\partial Y^2} - \frac{\varepsilon}{Pr} \frac{\partial^2 U}{\partial X^2} \right)
\]

(7)

\[
U \frac{\partial \theta}{\partial X} + V \frac{\partial \theta}{\partial Y} = \frac{1}{Re Pr} \frac{\partial^2 \theta}{\partial Y^2}
\]

(8)

\(\varepsilon\) takes value 0 for the case of an isothermal jet, and 1 when it is a heated jet.

The associated dimensionless boundary and inlet conditions become:

\(X = 0 ; \quad 0 < Y < 1: \quad V = 0 ; \quad \theta = 0 ;\)

Uniform profile : \(U = 1\),

Parabolic profile : \(U = (180)^{1/2} (1 - Y^2)\),

\(Y \geq 1 : U = 0 ; V = 0 ; \theta = 0 ;\)

\(X > 0 ; Y = 0 : U = 0 ; V = 0 ; \left( \frac{\partial \theta}{\partial Y} \right)_p = -1 ;\)

\(Y \to \infty : U = 0 ; \theta = 0 ;\)

III. NUMERICAL METHOD

Boundary layer equations, associated with the boundary conditions and the assumed velocity profiles at the exit of the nozzle were solved by a finite difference method. A shifted rectangular grid superimposed on the field of the flow is used, as shown on figure 1. This shifted grid allows a better approximation of the convective flow, a better evaluation of the pressure gradients as well as a numerical stabilization of the solution. The momentum and energy equations are discretized at nodes \((i + 1/2, j)\) whereas continuity equation is discretized at nodes \((i + 1/2, j + 1/2)\) and written as follows:

\[
\frac{U_{i+1,j} - U_{i,j}}{2\Delta X} + \frac{U_{i,j+1} - U_{i,j}}{2\Delta Y} + \frac{V_{i+1,j} - V_{i,j}}{2\Delta X} + \frac{V_{i,j+1} - V_{i,j}}{2\Delta Y} = 0 \quad (10)
\]

The transport equations of momentum and energy can be expressed in the following way:

\[
U \frac{\partial \psi}{\partial Y} + \frac{V}{\partial Y} = \frac{\partial}{\partial Y} \left( \frac{1}{Re} + \frac{\Gamma}{\psi} \right) \frac{\partial \psi}{\partial Y} + S_{\psi} = 0 \quad (11)
\]

and are respectively the diffusivity and the source term associated with the considered variable (U, V, and \(\psi\)). These equations are discretized at the nodes \((i + 1/2, j)\)and written as follows:

\[
U_{i+1/2,j+1/2} \left( \frac{\psi_{i+1,j+1/2} - \psi_{i-1,j+1/2}}{2\Delta Y} \right) + V_{i+1/2,j+1/2} \left( \frac{3\psi_{i+1,j+1/2} - 4\psi_{i,j+1/2} + \psi_{i-1,j+1/2}}{2\Delta Y} \right) = \frac{1}{Re + \Gamma_{i+1/2,j+1/2}} \left( \frac{\partial \psi_{i+1,j+1/2}}{\partial Y} - \frac{\partial \psi_{i-1,j+1/2}}{\partial Y} + \frac{\partial \psi_{i+1,j-1/2}}{\partial Y} - \frac{\partial \psi_{i-1,j-1/2}}{\partial Y} + \frac{\partial \psi_{i-1,j+1/2}}{\partial Y} - \frac{\partial \psi_{i-1,j-1/2}}{\partial Y} + \frac{\partial \psi_{i+1,j+1/2}}{\partial Y} - \frac{\partial \psi_{i-1,j+1/2}}{\partial Y} + \frac{\partial \psi_{i+1,j-1/2}}{\partial Y} - \frac{\partial \psi_{i-1,j-1/2}}{\partial Y} \right) + S_{\psi_{i+1/2,j}} \quad (12)
\]

A non-uniform grid is used in the flow (X) direction. The calculation step is very small in the vicinity of the nozzle (\(\Delta X = 10^{-6}\)) then it increases gradually as one moves away in the jet (\(\Delta X = 10^{-2}\)) in order to be able to go farther in the jet.

A non uniform grid according to the transverse direction is used. The distance between two successive nodes obeys the following relation : . Preliminary tests showed that by choosing a factor \(a=1.01\) and \(\Delta Y = 10^{-5}\), we obtain more than 50 points located in the viscous layer region. This condition ensures that the viscous layer adjacent to the wall is well taken into account. Far from the wall the distance between two successive nodes becomes constant and equal to \(\Delta Y=10^{-7}\). The grid used is rather small so that the use of the centred scheme (linear evolution of the variables between the nodes) ensures a satisfying stability.

Calculations begin on the leading edge of the plate \((X=Y=0)\) and are performed according to the fluid flow’s direction. The obtained discretized equations are solved by adopting a non linear Gauss Siedel method, which is a successive approximating method. This method, which was successfully used in former works \([16,17]\), was adopted for numerical stability reasons compared to methods using a non staggered grid discretization.

The convergence of the solution is reached when the relative change of U between two successive iterations is lower than \(10^{-7}\) for each node of the field. Additional iterations should not change the solution once convergence is reached.
IV. RESULTS AND DISCUSSION

In this section, the obtained results from numerical integration of the boundary layer equations coupled with the energy equation, governing laminar vertical wall jets, are presented and analyzed for different Prandtl numbers and for two velocity profiles at the nozzle exit: uniform profile or parabolic profile.

In order to generalize the study [18] and to validate it for different fluids, we considered a jet flow ejected tangentially to plane plate subjected to a constant heat flux density, in both forced and mixed convection regimes. The average Reynolds number at the exit of the nozzle is Re=1000. The fluid temperature at the nozzle exit is equal to that of the ambient conditions for the forced convection regime; the Froude number is equal to 225 for the other cases. The heat transfer between the jet and the plate is studied for Prandtl numbers ranging between 0.71 (case of the air) and 50.

A. Plane vertical wall jet in forced convection regime

1) Determination of the fictitious origin of the jet.

Initially, a dynamic study of the flow was established. Previous studies [2], [3],[8] showed that the behavior of the region located between the wall and the maximum velocity line differs notably from a traditional boundary layer flow because it evolves under an external flow action: It’s a disturbed boundary layer. A thorough study shows that from a certain distance of the nozzle exit, we can represent the external zone of the boundary layer by using a beam of iso-velocities in which is the jet maximum velocity in the considered section. This beam is defined in the area and it is made of linear curves passing by the same X-coordinate equal to \( X = -X_1 = -k_b \).

As it could be predicted, when the convection is forced, the variation of the Prandtl number has not any effect on the dynamics characteristics of the flow. Thus the position of the fictitious origin of the jet is about 210 times the thickness of the tube for all the treated cases (fig. 2).

2) Skin friction

The skin friction coefficient \( C_f = \frac{2\tau_p}{\rho u_m^2} \) is plotted according to the modified local Reynolds number
\[
\text{Re}_{x2} = \frac{u_m x_2}{\nu} = \frac{u_m (x + x_1)}{\nu}
\]
on figure 3.

No influence of the Prandtl number was signaled. All the fluid acts of the same manner for the same velocity profile at the nozzle exit. However, a difference is signaled in the jet zone between results for the uniform and parabolic velocity profiles. They became perfectly similar in the thermally established region. In fact, the influence of the emission conditions vanishes more one advances in the jet direction, the flow became controlled by the buoyancy forces. The curves tend towards the same asymptote. The negative slope of this curve returns to the fact that maximum velocity decreases for \( X \) growing, which induces a weaker velocity gradient and consequently a weaker shearing force. We notice that the linear decrease of this variable is according the relation:
\[
C_f = 1.28 \text{Re}^{-0.5}_{x2}
\]
(13)

This relation was established previously for the case of air (Pr=0.71) and various velocity at the nozzle exit [18].

3) Wall temperature

We represent on figure 4 the dimensionless wall temperature for the two velocity profiles at the nozzle exit and for the various Reynolds numbers. This figure shows that the velocity profiles only have an effect on the wall temperature in the region located close to the nozzle exit where the driving of ambient air by the jet is more significant with an initial uniform velocity profile. Further for ascending \( X \) values, the buoyancy forces become dominating, and a certain agreement between the temperature profiles are observed, which shows that the velocity profile have no longer influence on the wall temperature evolution in this region.
4) **Nusselt number**

In order to analyze the Prandtl number effect on the heat transfer between the jet and the wall subjected to a constant heat flux, we represent in figure 5 the streamwise evolution of the local Nusselt number for various fluids. This figure shows an increase in the heat transfer according to the longitudinal distance. For all the Prandtl numbers considered, the heat exchange is better as the Prandtl number increases. For the case of an uniform velocity profile at the nozzle exit, the heat transfer is more significant than that obtained with a parabolic profile. This difference increases when the distance from the nozzle increases and reaches its maximum in the intermediate zone. In the established zone this difference vanishes and becomes negligible. The fluid with low Prandtl number warms up more importantly, because for such fluids the effect of heat diffusion is higher.

\[ \text{Nu}_{12} = 0.61 \Pr^{0.33} \Re_{x2}^{0.5} \]  

\(\text{Nu}_{12}\) is the local Nusselt number, \(\Pr\) is the Prandtl number, \(\Re_{x2}\) is the modified Reynolds number.

Figure 6 shows the evolution of the modified Nusselt number according to the modified Reynolds number. These figure show similar evolutions as in the above paragraph. We note a linear evolution starting from a modified Reynolds number of about \(3 \times 10^5\) : this is the beginning of the thermally established region and for the various Prandtl numbers tested, the following correlation is founded

\[ \text{Nu}_{12} = 0.61 \Pr^{0.33} \Re_{x2}^{0.5} \]  

We have drawn on figure 7 the evolution of the average Nusselt numbers according to the Prandtl number for three plate lengths: \(L = 10, 100\) and \(1000\). For all the lengths considered and the two velocity profiles tested, we note that there is an increasing linear evolution. These figures confirm the previous interpretations according to which the velocity profile influences especially the shortest plates.
5) Colburn analogy

We give on figure 8 the Reynolds analogy factor. Where E; The obtained results show a light decreasing of the Reynolds analogy factor in the plume region (starting from about 1000 times the width of the nozzle) This shows an unsatisfactory agreement with the Colburn analogy; whereas it is valid in the case of a traditional boundary layer. This can be explained by the fact that the temperature dependence affect significantly the Nusselt number even when the Colburn factor remain unchanged.

B. Plane vertical wall jet in mixed convection

The variation of the Froude number has not any effect on the dynamics characteristics of the flow, as compared to the previous cases. Thus the position of the fictitious origin of the jet is about 210 times the thickness of the tube for all the treated cases. The thermal analysis of the emission conditions influence is treated for a heated flow (Fr = 225) and for various fluids.

1) Wall temperature

The longitudinal evolution of the modified wall temperature is shown on the figure 9 for various Prandtl numbers, for the two velocity profiles (uniform and parabolic) and for a Reynolds number equal to Re = 1000 and a Froude number equal to 225. In the region close to the nozzle, the inlet velocity profile influences the wall temperature evolution. Far from the ejection nozzle, a quite good agreement appears between the results obtained with the uniform velocity profile and the parabolic profile starting from the intermediate zone. This difference is negligible in the thermally established region.

For the heated jet, the buoyancy forces act at shorter distances.

2) Nusselt number

The evolution of the local Nusselt number is given in figure 10: in the vicinity of the nozzle exit, the inertia forces dominate the buoyancy forces so that the initial velocity profile influences in a significant degree the flow in this zone. The uniform profile ensures a more significant flow drive and, consequently, a more significant convective exchange and so a higher Nusselt number. For greater X distances, the effect of the initial velocity profile becomes negligible, as seen in the plume region.

The influence of the fluid heating increases as the flow moves away from the nozzle. In the thermal established region the heat exchange is more significant for all the fluids because the buoyancy forces that are responsible of the fluid behaviour in the plume are rather significant.

On figure 11 is shown the streamwise evolution of modified Nusselt number $\tilde{N}u_{x2} = \frac{X_2}{\theta_p(X)} = \frac{X + X_{1\tilde{p}}}{\theta_p(X)}$, according to the modified Reynolds number $Re_{x2} = \frac{\mu X_2}{\nu}$. One notes a linear evolution of the modified Nusselt number in the thermally established region.
established region.

![Graph showing streamwise evolution of the modified Nusselt number](image1)

**Fig. 10.** Streamwise evolution of the modified Nusselt number

This enables us to propose a correlation connecting the modified Nusselt number to the modified Reynolds number and the Prandtl number

\[ Nu_{x2} = 0.41 Pr^{0.33} Re_{x2}^{0.5} \]  

(15)

In order to generalize the study for various heat fluxes, other tests were carried. The previous correlation becomes:

\[ Nu_{x2} = 1.14 Pr^{0.33} Re_{x2}^{0.5} Gr_{X2}^{-0.012} \]  

(16)

3) **Colburn analogy**

We give on figure 11 the Reynolds analogy factor \( \frac{St}{Cf} \) for the different heated fluids. The obtained results show also a light decreasing in the plume region (starting from about 1000 times the width of the nozzle) for all the treated cases. This shows an unsatisfactory agreement with the Colburn analogy; even for heated fluids.

![Graph showing stream evolution of the Reynolds analogy factor](image2)

**Fig. 11.** Stream evolution of the Reynolds analogy factor

This influence is visible only in the zone of the jet: in this zone the flow is controlled by the inertia forces which imply a more significant drive of ambient fluid when an uniform velocity profile is used at the nozzle exit. In the intermediate zone where the buoyancy and inertia forces are of the same order the influence of the velocity profile vanishes. Further in plume region the buoyancy forces are dominating and so the velocity profile at the nozzle has no longer influence on the flow structure. We have also determined a correlation connecting the modified local Nusselt number according to the modified local Reynolds number. This correlation is valid in the region of the established mode for the two velocity profiles considered. \( Nu_{x2} = 0.61 Pr^{0.33} Re_{x2}^{0.5} \). For the case of mixed convection, the correlation becomes \( Nu_{x2} = 1.14 Pr^{0.33} Re_{x2}^{0.5} Gr_{X2}^{-0.012} \). It is noted that all the fluids react in the same manner with the variation of the inlet velocity profile, except that the fluid with a higher Prandtl number warms up more quickly than that with low Prandtl number. Thus we can have a more significant heat exchange with the plate.

**V. CONCLUSION**

In this work we have studied isothermal laminar vertical wall jets. The results relate to the influence of Prandtl number on the behaviour of the dynamics and thermal characteristics of the flow. The effect of the inlet conditions at the nozzle exit, and particularly the velocity profile is discussed. We have shown that the velocity profile at the nozzle exit affects the heat transfer between the jet and the heated wall

**REFERENCES**


