Some Remarks About Riemann-Liouville and Caputo Impulsive Fractional Calculus

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Abstract—This paper establishes some closed formulas for Riemann-Liouville impulsive fractional integral calculus and also for Riemann-Liouville and Caputo impulsive fractional derivatives.

Keywords—Riemann-Liouville fractional calculus, Caputo fractional derivative, Dirac delta, Distributional derivatives, High-order distributional derivatives.

I. Introduction

Fractional calculus has been used in a set of applications, mainly, to deal with modelling errors in differential equations and dynamic systems. There are also applications in Signal Processing and sampling and hold algorithms, [1-3]. Fractional integrals and derivatives can be of non-integer orders and even of complex order. This facilitates the description of some problems which are not easily described by ordinary calculus due to modelling errors, [1-5]. There are several approaches for the integral fractional calculus, the most popular ones being the Riemann-Liouville fractional integral. There is also a fractional Riemann-Liouville derivative. However, the well-known Caputo fractional derivative is less involved since the associated integral operator manipulates the derivatives of the primitive function under the integral symbol. This paper extends the basic fractional differ-integral calculus to fractional Riemann-Liouville impulsive fractional integral calculus and also by Basque Government by its support through the Spanish Ministry of Science and Innovation through Grant DPI2009-07197 and also by Basque Government by its support through Grant GIC07/143-IT-269-07.

II. Generalized Riemann-Liouville Fractional Integral

Let us denote the set of positive real numbers by $R_+ = \{ r \in R : r > 0 \}$ and left-sided and right-sided Lebesgue integrals, respectively, as:

$$\int_0^t g(\tau) d\tau : = \lim_{t \to x^-} \int_0^t g(\tau) d\tau$$

(the identification $x = x^-$ is used for all $x$ in order to simplify the notation), and

$$\int_0^+ g(\tau) d\tau : = \lim_{t \to x^+} \int_0^t g(\tau) d\tau$$

and $x = x^+$ implies that $g(\tau) = 0$ for all $\tau > x$.

Now, consider real functions $f, \tilde{f} : R_+ \rightarrow R$, such that

$$f(x) = \tilde{f}(x) + \sum_{i \in I} K_i \delta(x-x_i) = \tilde{f}(x) + \sum_{i \in I(x)} K_i \delta(x-x_i)$$

for each $x \in R_+$, fulfilling:

$$f(x) = \tilde{f}(x) + \sum_{i \in I(x)} K_i \delta(x-x_i) = \tilde{f}(x) + \sum_{i \in I(x)} K_i \delta(x-x_i)$$

Note that the existence of $\int_0^t (x-t)^\mu f(t) dt$ implies that

$$\int_0^t (x-t)^\mu f(t) dt = \int_0^t (x-t)^\mu \tilde{f}(t) dt$$

since $\int_0^t (x-t)^\mu \tilde{f}(t) dt$ exists, and that of

$$\int_0^+ (x-t)^\mu f(t) dt = \int_0^+ (x-t)^\mu \tilde{f}(t) dt + (x-x_i)^{\mu-1} \tilde{f}(x_i)$$

for $x_i \in I(x)$.
Theorem 2.1. The extended fractional Riemann- Liouville integrals by considering impulsive functions are defined for any fixed order \( \mu \in \mathbb{R}_+ \) and all \( x \in \mathbb{R}_+ \) by

\[
\left( J^{\mu} f \right)(x) := \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt
\]

1. \( \int_0^0 (x-t)^{\mu-1} f(t) dt = 0 \) and if \( x \notin \text{IMP} \), then:

\[
\left( J^{\mu} f \right)(x^+) = \left( J^{\mu} f \right)(x).
\]

2. \( \int_0^0 (x-t)^{\mu-1} f(t) dt = 0 \) and if \( x \notin \text{IMP} \), then:

\[
\left( J^{\mu} f \right)(x^+) = \left( J^{\mu} f \right)(x).
\]

Theorem 3.1. Assume that \( f \in C^{m-2} \) and \( f(eR_+) \) and its \( m-\)th derivative exists everywhere in \( R_+ \). Then, the Caputo fractional derivative of order \( \mu \geq 0 \) with \( m-1 \leq \mu \leq m \) is defined by

\[
\left( D^{\mu} f \right)(x) := \frac{d^m}{dx^m} \left( \int_0^x (x-t)^{m-\mu-1} f(t) dt \right)
\]

\[
= \frac{1}{\Gamma(m-\mu)} \left( \frac{d}{dx} \right)^m \left( \int_0^x (x-t)^{m-\mu-1} f(t) dt \right)
\]

The following particular cases follow from this formula for \( \mu = m-1 \):

(a) \( \mu = -1; m = 0 \) yields \( \left( D^{-1} f \right)(x) = \int_0^x f(t) dt \) which is the standard integral of the function \( f \). This case does not verify the “derivative constraint” 0 \( \leq m-1 \leq \mu \leq m \), leading to an integral result.

(b) \( \mu = 0; m = 1 \) yields \( \left( D^0 f \right)(x) = f(x) \) which so that \( D^0 f \) is the identity operator.

(c) \( \mu = 1; m = 2 \) yields \( \left( D^1 f \right)(x) = f'(x) \)

(d) \( \mu = 2; m = 3 \) yields \( \left( D^2 f \right)(x) = f''(x) \) which is the standard first- derivative of the function \( f \).
\[ + \frac{1}{\Gamma(m-\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau \]

If \( f \in PC^k(\mathbb{R}^+, \mathbb{R}) \) with \( f^{(k)}(x) \) being discontinuous in the first class then \( f^{(m-1)}(x) = \delta(j(x)) \) with \( j(x) = m-1-k(x) \), one uses to obtain the right value of (8) the perhaps high-order distributional derivatives formula:

\[ \left[ f^{(m-1)}(x) - f^{(m-1)}(x) \right] \frac{(-1)^k k!}{x^k} \int_0^t (t-\tau)^{m-1-k} f(\tau) d\tau \delta(0) = 0 \quad (9) \]

to yield

\[ \left( D^\mu f \right)(x) = \frac{1}{\Gamma(m-\mu)} \int_0^t (t-\tau)^{m-1} f(\tau) d\tau \]

If \( m = m-1 \) then

\[ \left( D^{m-1} f \right)(x) = f^{(m-1)}(x) \]

provided that \( \int_0^t (t-\tau)^{m-1} f(\tau) d\tau \) exists for \( x \in \mathbb{R}^+ \) which is guaranteed if \( f(t) \) is Lebesgue-integrable on \( \mathbb{R}^+ \) and \( f \in C^{m-1}(\mathbb{R}^+, \mathbb{R}) \) as for the extended Riemann–Liouville fractional integral, one gets:

\[ \left( D^{m-1} f \right)(x) = \frac{1}{\Gamma(m-\mu)} \int_0^T (t-\tau)^{m-1} f(\tau) d\tau \]

IV. GENERALIZED CAPUTO FRACTIONAL DERIVATIVE

Assume that \( f \in C^m(\mathbb{R}^+, \mathbb{R}) \) and its \( m-th \) derivative exists everywhere in \( \mathbb{R}^+ \). Then, the Caputo fractional derivative of order \( \mu \geq 0 \) with \( m-1 \leq \mu \in \mathbb{R}^+ < m \), \( m \in \mathbb{Z}^+ \) is for any \( x \in \mathbb{R}^+ \):

\[ \left( D^{\mu} f \right)(x) = \frac{1}{\Gamma(m-\mu)} \int_0^T (t-\tau)^{m-1} f(\tau) d\tau \]

The following particular cases occur with \( \mu = m-1 \) leading to

\[ \left( D^{m-1} f \right)(x) = \frac{1}{\Gamma(m-\mu)} \int_0^T (t-\tau)^{m-1} f(\tau) d\tau \]

(a) \( \mu = 1; m=0 \) yields

\[ \left( D^{1} f \right)(x) = \frac{1}{\Gamma(1-\mu)} \int_0^T (t-\tau)^{1-1} f(\tau) d\tau \]

(b) \( \mu = 0; m=1 \) yields

\[ \left( D^{0} f \right)(x) = f(0) - f(0) = f(0) - f(0) = f(0) - f(0) \]

(c) \( \mu = 1; m=2 \) yields

\[ \left( D^{1} f \right)(x) = f(1) - f(1) = f(1) - f(1) \]

(d) \( \mu = 2; m=3 \) yields

\[ \left( D^{2} f \right)(x) = f(2) - f(2) = f(2) - f(2) \]

We can extend the above formula to real functions with impulsive \( m-th \) derivative as follows. Assume that \( f \in C^m(\mathbb{R}^+, \mathbb{R}) \) with bounded piecewise \( (m-1)-th \)
derivative existing everywhere in $\mathbb{R}_+$ and

$$f^{(m)}(x) = \frac{d^m f(x)}{dx^m}$$ being impulsive with

$$f^{(m)}(x_i) = K_i \delta(0) = \left( f^{(m-1)}(x_i^-) - f^{(m-1)}(x_i^+) \right) \delta(0)$$

for some $x_i \in \text{IMP}$, equivalently, $\forall i \in I(\infty)$, at the eventual discontinuity points $x_i > 0$ at the impulsive set $\text{IMP} = \bigcup_{x \in \mathbb{R}_+} \text{IMP}(x)$, where the partial impulsive sets are re-defined as follows:

$$\text{IMP}(x) := \left\{ x \in \mathbb{R}_+ \mid f^{(m-1)}(x_i^-) - f^{(m-1)}(x_i^+) = K_i, x_i < x \right\} \subseteq \text{IMP}(x^+)$$

$$\text{IMP}(x^+) := \left\{ x \in \mathbb{R}_+ \mid f^{(m-1)}(x_i^-) - f^{(m-1)}(x_i^+) = K_i, x_i \leq x^+ \right\} \subseteq \text{IMP}(x^+)$$

(16)

(17)

Now, consider $f \in C^{m-1}(0, \infty)$ with $f^{(m)}(x) = \frac{d^m f(x)}{dx^m}$ being almost everywhere piecewise continuous in $\mathbb{R}_+$ except possibly on a non-empty discrete impulsive set $\text{IMP}$. Define a non-impulsive real function $\tilde{f} : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined as $\tilde{f}(m)(x) = f^{(m)}(x)$ for $x \in \mathbb{R}_+ \setminus \text{IMP}$, and $f^{(m)}(x_i) = \tilde{f}(m)(x_i)$ for $x_i \in \text{IMP}$ with $\tilde{f}(m)(x^+) = f^{(m)}(x)$; $x \in \text{IMP}^+$ defined being bounded arbitrary (for instance, zero) if $x \in \text{IMP}$. Through a similar reasoning as that used for Riemann-Liouville fractional integral by replacing the function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ by its m-th derivative, one obtains the following result:

**Theorem 4.1.** The Caputo fractional derivative of order $\mu \in \mathbb{R}_+$ satisfying $m-1 < \mu \leq m$; $m \in \mathbb{Z}_+$ and all $x \in \mathbb{R}_+$ is given below:

$$\left(D^\mu f\right)(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} \int_{x_i^-}^{x_i^+} (x-t)^{m-\mu-1} \tilde{f}(m)(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} \int_{x_i^+}^{x_i^+} (x-t)^{m-\mu-1} \tilde{f}(m)(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1} f^{(m-1)}(x_i^-) \delta(x-x_i)$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1} \tilde{f}(m-1)(x_i^-) \delta(x-x_i)$$

$$= \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} \int_{x_i^-}^{x_i^+} (x-t)^{m-\mu-1} f^{(m)(t)} \delta(x-x_i)$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1} f^{(m-1)}(x_i^-) \delta(x-x_i)$$

(18)

(19)

where $n : \text{IMP} \rightarrow \mathbb{Z}_+$ is a discrete function defined by $n(x) = \text{card } I(x) = \text{card } \text{IMP}(x)$.

Note that if $x \in \text{IMP}$ then

$$\left(D^\mu f\right)(x^+) := \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+)} \int_{x_i^-}^{x_i^+} (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+)} \int_{x_i^-}^{x_i^+} f^{(m-1)}(x_i^-) \delta(x-x_i)$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+)} (x-x_i)^{m-\mu-1} f^{(m-1)}(x_i^-) \delta(x-x_i)$$

and if $x \in \text{IMP}$, since $I(x^+)=I(x)$, then

$$\left(D^\mu f\right)(x^+) = \left(D^\mu f\right)(x^+)$$.

The above formalism applies when $f^{(m)} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is piecewise continuous with isolated first-class discontinuity points, that is $f \in PC^{m-1} (\mathbb{R}_+, \mathbb{R})$ implying that $f \in C^{m-2} (\mathbb{R}_+ \setminus \text{IMP}, \mathbb{R})$. A more general situation arises when the discontinuities can point-wise arise for points of the function itself of for any successive derivative up-till order m. This would lead to a more general description than that given as follows. Define partial sets of positive integers as $\overline{I} := \{1, 2, \ldots, k\}$

Assume that $f \in PC^j (\mathbb{R}_+, \mathbb{R})$ and $x$ is a discontinuity point of first class of $f^{(j)}(x)$ for some $j \in m-1 \cup \{0\}$.

Then, $f^{(j+i)}(x)$ are impulsive for $i \in m-1$ of high order being increasing with $\ell$. Define the $(j+i)-th$ impulsive sets of the function $f$ on $(0, x) \subset \mathbb{R}$ as:

$$\text{IMP}_{j+i}(x) := \left\{ x \in \mathbb{R}_+ \mid 0 < f^{(j)}(x^+) - f^{(j)}(x^-) \right\}$$

$$j \in m-1 \cup \{0\}, x \in \mathbb{R}_+$$

(20)

This leads directly the definition of the following impulsive sets:

$$\text{IMP}_{j+i} := \left\{ x \in \mathbb{R}_+ \mid 0 < f^{(j)}(x^+) - f^{(j)}(x^-) \right\}$$

$$j \in m-1 \cup \{0\}$$

(21)

$$\text{IMP}_{j+i} := \left\{ x \in \mathbb{R}_+ \mid 0 < f^{(j)}(x^+) - f^{(j)}(x^-) \right\}$$

$$j \in m-1 \cup \{0\}$$

(22)

which can be empty. Thus, if $z \in \text{IMP}_{j+i}$ then $f^{(j)}(x^+) = f^{(j)}(x^-)$ exists with identical left and right
limits, \[ f^{(j)}(x^+)- f^{(j)}(x)= K = K(x) \neq 0 \quad \text{and} \quad f^{(j)}(x)= K \delta(0) \] with successive higher-order derivatives represented by higher-order Dirac distributional derivatives.

The above definitions yield directly the following simple results:

**Assertion 5.2.** \( x \in \text{IMP} \Rightarrow x \in \text{IMP}_j \) for a unique \( j=j(x) \in m \).

**Proof:** Proceed by contradiction. Assume that \( x \in \{\text{IMP}_{i+1} \cap \text{IMP}_{j+1}\} \) for \( i, j \neq i \) \( \in \mathbb{m}-1 \cup \{0\}. \) Then:

\[
0 < \left| f^{(j)}(x^+)- f^{(j)}(x) \right|< \infty \quad ; \quad 0 < \left| f^{(j)}(x^+)- f^{(j)}(x) \right|< \infty
\]

Assume with no loss of generality that \( j=i+k > i \) for some \( k \leq m-i \in Z_+ \). Then,

\[
f^{(j)}(x^+)- f^{(j)}(x) = \left(\frac{-1}{x^k}\right) f^{(j)}(x^+)- f^{(j)}(x) \delta(0) = \infty
\]

with \( x \in \mathbb{R}_+ \). If \( f^{(j)}(x^+)- f^{(j)}(x) \neq 0 \) which contradicts

\[
0 < \left| f^{(j)}(x^+)- f^{(j)}(x) \right|< \infty \quad \text{so that} \quad i=j. \quad \Box
\]

**Assertion 5.3.** \( x \in \text{IMP} \Rightarrow \exists \text{ a unique } j=j(x) = \max_{i \in \mathbb{m}} \left| f^{(i)}(x^+)- f^{(i)}(x-1) \right|< \infty \)

Furthermore, such a unique \( j=j(x) \) satisfies

\[
f^{(j-1)}(x^+)- f^{(j-1)}(x) \geq 0.
\]

**Proof:** The existence is direct by contradiction. If \( \exists \ j = j(x) \in \mathbb{m}-1 \cup \{0\} \) such that

\[
f^{(j)}(x^+)- f^{(j)}(x) \geq \infty \quad \text{then} \quad x \notin \text{IMP} \). Now, assume there exist two nonnegative integers \( i=j(x)= \max_{i \in \mathbb{m}} \left| f^{(i)}(x^+)- f^{(i)}(x-1) \right|< \infty \) and

\[
j=j(x)=i+k = \max_{i \in \mathbb{m}} \left| f^{(i+k)}(x^+)- f^{(i+k)}(x) \right|< \infty \quad \text{for some} \quad k \leq m-i. \]

But for \( x > 0 \),

\[
\infty = \left(\frac{-1}{x^k}\right) f^{(i)}(x^+)- f^{(i)}(x) \delta(0) = \left| f^{(i+k)}(x^+)- f^{(i+k)}(x) \right|< \infty
\]

which is a contradiction. Then, \( x \in \text{IMP}_j \Rightarrow \exists \ j=j(x) = \max_{i \in \mathbb{m}} \left| f^{(i)}(x^+)- f^{(i)}(x) \right|< \infty \)

which is unique. Also, from the definition of the impulsive sets \( \text{IMP}_j(x) \).
\[ f^{(m-1)}(x) = \frac{1}{(m-\mu)} \int_0^t (x-t)^{m-\mu-1} f^{(m)}(t) \, dt \]

\[ f^{(m-1)}(x^+) = \frac{1}{(m-\mu)} \sum_{i \in J(x^+)} \int_{x_i^-}^{x_i^+} (x-t)^{m-\mu-1} \tilde{f}^{(m)}(t) \, dt \]

\[ + \frac{1}{(m-\mu)} \sum_{i \in J(x)} \int_{x_i^-}^{x_i} (x-t)^{m-\mu-1} \tilde{f}^{(m)}(t) \, dt \]

\[ + \frac{1}{(m-\mu)} \sum_{i \in J(x)} \int_{x_i}^{x_i^+} (x-t)^{m-\mu-1} \tilde{f}^{(m)}(t) \, dt \]

\[ \times \left( \frac{m-j(x_i)-1}{(x-x_i)^{m-j(x_i)+1}} (f(j(x_i))(x_i^+)-f(j(x_i))(x_i^-)) \right) \]

\[ \left( D_x^\mu f \right)(x^+) = \frac{1}{(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t) \, dt \]

\[ + \frac{1}{(m-\mu)} \sum_{i \in J(x^+)} \int_{x_i^-}^{x_i^+} (x-t)^{m-\mu-1} \tilde{f}^{(m)}(t) \, dt \]

\[ + \frac{1}{(m-\mu)} \sum_{i \in J(x)} \int_{x_i^-}^{x_i} (x-t)^{m-\mu-1} \tilde{f}^{(m)}(t) \, dt \]

\[ + \frac{1}{(m-\mu)} \sum_{i \in J(x)} \int_{x_i}^{x_i^+} (x-t)^{m-\mu-1} \tilde{f}^{(m)}(t) \, dt \]

\[ \times \left( \frac{m-j(x_i)-1}{(x-x_i)^{m-j(x_i)+1}} (f(j(x_i))(x_i^+)-f(j(x_i))(x_i^-)) \right) \]

\[ \left( D_x^\mu f \right)(x^+) = \infty \text{ if } x = x_i \in \text{IMP} \]

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