Some Remarks About Riemann-Liouville and Caputo Impulsive Fractional Calculus

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Abstract—This paper establishes some closed formulas for Riemann-Liouville impulsive fractional integral calculus and also for Riemann-Liouville and Caputo impulsive fractional derivatives.

Keywords—Riemann-Liouville fractional calculus, Caputo fractional derivative, Dirac delta, Distributional derivatives, High-order distributional derivatives.

I. INTRODUCTION

Fractional calculus has been used in a set of applications, mainly, to deal with modeling errors in differential equations and dynamic systems. There are also applications in Signal Processing and sampling and hold algorithms, [1-3]. Fractional integrals and derivatives can be of non-integer orders and even of complex order. This facilitates the description of some problems which are not easily described by ordinary calculus due to modeling errors, [1-5]. There are several approaches for the integral fractional calculus, the most popular ones being the Riemann-Liouville fractional integral. There is also a fractional Riemann-Liouville derivative. However, the well-known Caputo fractional derivative are less involved since the associated integral operator manipulates the derivatives of the primitive function under the integral symbol. This paper extends the basic fractional differ-integral calculus to impulsive functions described through the use of Dirac distributions and Dirac distributional derivatives, [5], of real fractional orders. In the general case, it is admitted a presence of infinitely many impulsive terms at certain isolated point of the relevant function domains. Control Theory topics in [6-9] could be reformulated under the fractional formalism considered in this paper.

II. GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL

Let us denote the set of positive real numbers by $\mathbb{R}_+ = \{ r \in \mathbb{R} : r > 0 \}$ and left-sided and right-sided Lebesque integrals, respectively, as:

$$\int_0^x g(t) \, dt := \lim_{\tau \to x^-} \int_0^\tau g(\tau) \, d\tau \quad \text{(the identification)}$$

$x = x^-$ is used for all $x$ in order to simplify the notation, and

$$\int_0^x g(t) \, dt := \lim_{\tau \to x^-} \int_0^\tau g(\tau) \, d\tau$$

Now, consider real functions $f, \tilde{f} : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\int_0^x (x-t)^{-\mu - 1} f(t) \, dt \quad \forall x \in \mathbb{R}_+,$$

fulfilling:

$$f(x) = \tilde{f}(x) + \sum_{i \in \mathbb{I}} K_i \delta(x-x_i) = \tilde{f}(x) + \sum_{i \in \mathbb{I}} K_i \delta(x-x_i)$$

The Dirac delta function $\delta(x)$ denotes the Dirac delta distribution, $K_i \delta(0) = f(x_i)$ for $K_i \in \mathbb{R}$, $i \in \mathbb{I}$ and $\mathbb{I}$ the indexing set of index set $I(x)$ is the whole impulsive set defined via empty or non-empty) partial impulsive strictly ordered denumerable sets:

$$I(x) := \{ i \in \mathbb{Z}_0^+ : x_i \in \mathbb{I}(x) \} I(x) \subset \mathbb{Z}_+ \quad \text{(indexing set)}$$

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The existence of infinitely many impulsive terms at certain isolated point of the relevant function domains. Control Theory topics in [6-9] could be reformulated under the fractional formalism considered in this paper.

Note that the existence of $\int_0^x (x-t)^{-\mu - 1} f(t) \, dt$ implies that of $\int_0^x (x-t)^{-\mu - 1} \tilde{f}(t) \, dt$ if $x \in \mathbb{I}(x)$, since $\int_0^x (x-t)^{-\mu - 1} \tilde{f}(t) \, dt$ exists, and that of $\int_0^x (x-t)^{-\mu - 1} f(t) \, dt = \int_0^x (x-t)^{-\mu - 1} \tilde{f}(t) \, dt$ if $x \in \mathbb{I}(x)$, since $\int_0^x (x-t)^{-\mu - 1} \tilde{f}(t) \, dt$ exists, and that of

$$\int_0^x (x-t)^{-\mu - 1} f(t) \, dt = \int_0^x (x-t)^{-\mu - 1} \tilde{f}(t) \, dt + (x-x_i)^{-\mu - 1} \tilde{f}(x_i)$$

if $x \in \mathbb{I}(x)$

$$\int_0^x (x-t)^{-\mu - 1} f(t) \, dt = \int_0^x (x-t)^{-\mu - 1} \tilde{f}(t) \, dt + (x-x_i)^{-\mu - 1} \tilde{f}(x_i)$$

if $x \in \mathbb{I}(x)$
Theorem 2.1. The extended fractional Riemann-Liouville integrals by considering impulsive functions are defined for any fixed order \( \mu \in \mathbb{R} \) and all \( x \in \mathbb{R} \) by

\[
\left( J^\mu f \right)(x) := \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt
\]

\[
= \frac{1}{\Gamma(\mu)} \left( \int_0^x (x-t)^{\mu-1} f(t) dt + \sum_{i \in I(x) \cup \{0\}} (x-x_i)^{\mu-1} f(x_i^-) - f(x_i) \right)
\]

\[
+ \frac{1}{\Gamma(\mu)} \sum_{i \in I(x)} (x-x_i)^{\mu-1} f(x_i^-) - f(x_i)
\]

where \( I: \mathbb{R}_+ \to \mathbb{R}_+ \) is the \( I \) - function , [1-5] and \( n: \text{IMP} \to \mathbb{Z}_+ \) is defined by

\[
n(x) = \text{card \, I}(x) = \text{card \, IMP}(x).
\]

Note that if \( x \in \text{IMP} \) then

\[
\left( J^\mu f \right)(x) = \frac{1}{\Gamma(\mu)} \sum_{i \in I(x) \cup \{0\}} \int_{x_i}^{x_i^+} (x-t)^{\mu-1} f(t) dt
\]

\[
+ \frac{1}{\Gamma(\mu)} \sum_{i \in I(x)} (x-x_i)^{\mu-1} f(x_i^-) - f(x_i)
\]

and if \( x \notin \text{IMP} \), since \( I(x^+) = I(x) \), then

\[
\left( J^\mu f \right)(x) = \frac{1}{\Gamma(\mu)} \sum_{i \in I(x)} \int_{x_i}^{x_i^+} (x-t)^{\mu-1} f(t) dt
\]

The following particular cases follow from this formula for \( \mu=m-1 \):

(a) \( \mu = -1; m = 0 \) yields \( \left( D^{-1} f \right)(x) = \int_0^x f(t) dt \) which is the standard integral of the function \( f \). This case does not verify the “derivative constraint” \( 0 \leq m-1 \leq \mu_{\mathbb{R}_+} < m \) leading to an integral result.

(b) \( \mu = 0; m = 1 \) yields \( \left( D^0 f \right)(x) = f(x) \) which so that \( D^0 f \) is the identity operator.

(c) \( \mu = 1; m = 2 \) yields \( \left( D^1 f \right)(x) = f'(x) \).

(d) \( \mu = 2; m = 3 \) yields \( \left( D^2 f \right)(x) = f''(x) \) which is the standard first-derivative of the function \( f \).

Compared to the parallel cases with the Caputo fractional derivative, note that the Riemann-Liouville fractional derivative, compared to the Caputo corresponding one, does not depend on the conditions at zero of the function and its derivatives. Define the Kronecker delta \( \delta_{ab} \) of any pair of real numbers \( (a,b) \) as \( \delta_{aa} = 1 \) if \( a = b \) and \( \delta_{ab} = 0 \) if \( a \neq b \) and then evaluate recursively the Riemann–Liouville fractional derivative of order \( \mu \geq 0 \) from the above formula by using Leibniz’s differentiation rule by noting that, since \( \mu \neq m-j \) for \( \mu \in \mathbb{Z}_+ \), the only differential part corresponding to the differentiation of the integrand is non zero for \( j > m - \mu \). This yields the following result:

Theorem 3.1. Assume that \( f \in C^{m-1} (\mathbb{R}_+, \mathbb{R}) \) and its \( m-1 \) derivative exists everywhere in \( \mathbb{R}_+ \) and that \( f(t) \) is integrable on \( \mathbb{R}_+ \), then:

\[
\left( D^\mu f \right)(x) = \frac{1}{\Gamma(m-\mu)} \left( \frac{d}{dx} \right)^m \left( \int_0^x (x-t)^{m-\mu-1} f(t) dt \right)
\]

\[
= \frac{1}{\Gamma(m-\mu)} \left( \frac{d}{dx} \right)^{m-1} \left[ \int_0^x (x-t)^{m-\mu-2} f(t) dt + f(x) \delta(\mu, m-1) \right]
\]

and if \( x \notin \text{IMP} \), since \( I(x^+) = I(x) \), then

\[
\left( J^\mu f \right)(x) = \left( J^\mu f \right)(x).
\]
If \( f \in PC^k(R_+, \mathbb{R}) \) with \( f^{(k)}(x) \) being discontinuous of first class then \( f^{(m-1)}(x) = \theta^\mu(j(x)) \) with \( j(x) = m-1 - k(x) \), one uses to obtain the right value of (8) the perhaps high-order distributional derivatives formula:

\[
\frac{f^{(m-1)}(x^+) - f^{(m-1)}(x)}{x^+} = \frac{1}{x^+} \int_{x^+}^{x} f^{(m-4)}(y) f^{(m-1-k)}(y) \delta(0^+) \delta(\mu, m-1) dx
\]

\[
+ \sum_{i=1}^{m-1} \sum_{j=0}^{m-i} [j - \mu] \int_{x^+}^{x} f^{(j)}(x) \delta(\mu, m-i) dx
\]

\[
+ \sum_{j=0}^{m-1} [j - \mu] \left( \int_{x^+}^{x} f^{(j)}(x) \delta(\mu, m-i) dx \right)
\]

(10)

If \( \mu = m-1 \) then

\[
(D^{m-1}f)(x) = f^{(m-1)}(x) + \sum_{j=0}^{m-1} [j - \mu] \left( \int_{x^+}^{x} f^{(j)}(x) \delta(\mu, m-i) dx \right)
\]

(11)

provided that \( \left( \int_{x^+}^{x} f^{(j)}(x) \delta(\mu, m-i) dx \right) \) exists for \( x \in R_+ \) (which is guaranteed if \( f^{(j)}(x) \) is Lebesgue-integrable on \( R_+ \), \( f \in C^{m-2}(R_+, \mathbb{R}) \)) and \( f^{m-1} \) exists everywhere in \( R_+ \). The correction (10) applies when the derivative does not exist.

\[
\frac{D^{m-1}f(x)}{x^+} = \frac{f^{(m-1)}(x)}{x^+} - \sum_{j=0}^{m-1} [j - \mu] \left( \int_{x^+}^{x} f^{(j)}(x) \delta(\mu, m-i) dx \right)
\]

(12)

(13)

IV. GENERALIZED CAPUTO FRACTIONAL DERIVATIVE

Assume that \( f \in C^{m-1}(R_+, \mathbb{R}) \) and its \( m-1 \) derivative exists everywhere in \( R_+ \). Then, the Caputo fractional derivative of order \( \mu \geq 0 \) with \( m-1 \leq \mu \langle R_+ \rangle < m \), \( m \in \mathbb{Z}^+ \) is for any \( x \in R_+ \):

\[
(D^{\mu}f)(x) = \left( \frac{D^{m-1}f(x)}{x^+} \right)
\]

(14)

The following particular cases occur with \( \mu = m-1 \) leading to

\[
(D^{m-1}f)(x) = \left( \frac{D^{m-1}f(x)}{x^+} \right)
\]

(15)

(a) \( \mu = -1; m = 0 \) yields \( (D^{\mu}f)(x) = f^{(-1)}(x) - f^{(-1)}(x^+) \) which is an integral result \( f \). Note that this case does not verify the “derivative constraint” \( 0 \leq \mu \langle R_+ \rangle < m \) leading to an integral result.

(b) \( \mu = 0; m = 1 \) yields

\[
(D^{\mu}f)(x) = f^{(0)}(x) - f^{(0)}(x^+) = f(x) - f(x^+)
\]

(c) \( \mu = 1; m = 2 \) yields

\[
(D^{\mu}f)(x) = f^{(1)}(x) - f^{(1)}(x^+) = f(x) - f(x^+)
\]

(d) \( \mu = 2; m = 3 \) yields

\[
(D^{\mu}f)(x) = f^{(2)}(x) - f^{(2)}(x^+) = f(x) - f(x^+)
\]

We can extend the above formula to real functions with \( m-\text{th} \) derivative as follows. Assume that \( f \in C^{m-2}(R_+, \mathbb{R}) \) with bounded piecewise \( (m-1)-\text{th} \)
derivative existing everywhere in $\mathbb{R}^+_+$ for some $x \in \text{IMP}_x$ equivalently, $\forall i \in I(\infty)$, at the eventual discontinuity points $x_i > 0$ at the impulsive set $\text{IMP}_x := \bigcup x \in \text{IMP}_x$, where the partial impulsive sets are re-defined as follows:

$$\text{IMP}_x := [x, \in \mathbb{R}^+_+, f^{(m-1)}(x) = f^{(m-1)}(x) = K_i, x_i < x \in \text{IMP}_x$$

$$\text{IMP}_x := \{x, \in \mathbb{R}^+_+, f^{(m-1)}(x) = K_i, x_i \leq x \in \text{IMP}_x$$

Now, consider $f \in C^{m-1}(0, \infty)$ with

$$f^{(m)}(x) = \frac{d^m f}{dx^m}$$

continuous in $\mathbb{R}^+_+$ except possibly on a non-empty discrete impulsive set $\text{IMP}$. Define a non-impulsive real function

$$\tilde{f} : \mathbb{R}^+_+ \rightarrow \mathbb{R}$$

defined as $\tilde{f}^{(m)}(x) = f^{(m)}(x)$ for $x \in \mathbb{R}^+_+$, and $f^{(m)}(x) = \tilde{f}^{(m)}(x) = K_i \delta(0)$ for $x \in \text{IMP}_x$ with

$$\tilde{f}^{(m)}(x) = f^{(m)}(x) ; x \in \text{IMP}_x \text{ defined being bounded arbitrary (for instance, zero) if } x \in \text{IMP}.$$

Through a similar reasoning as that used for Riemann-Liouville fractional integral by replacing the function $f: \mathbb{R}^+_+ \rightarrow \mathbb{R}$ by its m-derivative, one obtains the following result:

**Theorem 4.1.** The Caputo fractional derivative of order $\mu \in \mathbb{R}_+$ satisfying $m-1 < \mu \leq m ; m \in \mathbb{Z}_+$ and all $x \in \mathbb{R}_+$ is given below:

$$\left(D^\mu f\right)(x):= \frac{1}{\Gamma(\mu - m)} \int_0^x (x-t)^{\mu-m-1} f^{(m)}(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \int_{i \in I(x)} (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1} f^{(m)}(x_i)\delta(x-x_i)$$

(18)

$$\left(D^\mu f\right)(x)^2:= \frac{1}{\Gamma(\mu - m)} \int_0^x (x-t)^{\mu-m-1} f^{(m)}(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \int_{i \in I(x)} (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1} f^{(m)}(x_i)\delta(x-x_i)$$

(19)

where $n: \text{IMP} \rightarrow \mathbb{Z}_+$ is a discrete function defined by

$$n(x) = \text{card I}(x) = \text{card IMP}(x).$$

Now that if $x \in \text{IMP}$ then

$$\left(D^\mu f\right)(x)^2 := \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} \int_{0}^{x_i} (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} \int_{0}^{x_i} (x-x_i)^{m-\mu-1} f^{(m)}(x_i)\delta(x-x_i)$$

$$\left(D^\mu f\right)(x)^2 := \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} \int_{0}^{x_i} (x-t)^{m-\mu-1} f^{(m)}(t)dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} \int_{0}^{x_i} (x-x_i)^{m-\mu-1} f^{(m)}(x_i)\delta(x-x_i)$$

and if $x \in \text{IMP}$, since $I(x^+) = I(x)$, then

$$\left(D^\mu f\right)(x)^2 := \left(D^\mu f\right)(x)^2.$$
limits, \( f^{(i)}(x^+) - f^{(i)}(x^-) = K = K(x) \neq 0 \) and
\[ f^{(i)}(x^-) = K\delta(0) \] with successive higher-order derivatives represented by higher-order Dirac distributional derivatives.

The above definitions yield directly the following simple results:

**Assertion 5.2.** \( x \in \text{IMP} \Rightarrow x \in \text{IMP}_j \) for a unique \( j = j(x) \in \mathbb{m} \).

**Proof:** Proceed by contradiction. Assume that \( x \in \{ \text{IMP}_{i+1} \cap \text{IMP}_{j+1} \} \) for \( i, j \neq i \in \mathbb{m} - 1 \cup \{0\} \). Then:
\[ 0 < \left| f^{(i)}(x^+) - f^{(i)}(x^-) \right| < \infty \]
\[ 0 < \left| f^{(j)}(x^+) - f^{(j)}(x^-) \right| < \infty \]

Assume with no loss of generality that \( j = i + k > i \) for some \( k \leq m - i - 1 \in \mathbb{Z}_+ \). Then,
\[ \left| f^{(i)}(x^+) - f^{(j)}(x^-) \right| = \left| \frac{(-1)^i k!}{x^i} f^{(i)}(x) - f^{(j)}(x) \right| = \left| \delta(0) \right| = \infty \]
with \( x \in \mathbb{R}^+ \). If \( \left| f^{(i)}(x^+) - f^{(j)}(x^-) \right| \neq 0 \) which contradicts
\[ 0 < \left| f^{(j)}(x^+) - f^{(j)}(x^-) \right| < \infty \] so that \( i = j \). \( \Box \)

**Assertion 5.3.** \( x \in \text{IMP} \Rightarrow \)
\[ x \in \text{IMP}_j \Leftrightarrow \text{there exist two nonnegative integers } i \neq j \text{ and } i + k > i \text{ for some } k \leq m - i - 1 \in \mathbb{Z}_+ \text{ such that} \]
\[ f^{(i)}(x^+) - f^{(j)}(x^-) < \infty \text{ then } x \notin \text{IMP}. \] Now, assume there exist two nonnegative integers
\[ i = i(x) = \max_{i \in \mathbb{m}} \left| f^{(i-1)}(x^+) - f^{(i-1)}(x^-) \right| < \infty \]
and
\[ j = j(x) = i + k = \max_{i \in \mathbb{m}} \left| f^{(i+k-1)}(x^+) - f^{(i+k-1)}(x^-) \right| < \infty \] for some \( k \leq m - i - 1 \in \mathbb{Z}_+ \). But for \( x > 0 \),
\[ \infty = \frac{(-1)^i k!}{x^i} \left| f^{(i-1)}(x^+) - f^{(i-1)}(x^-) \right| \delta(0) \]
which is a contradiction. Then,
\[ x \in \text{IMP}_j \Rightarrow \exists j = j(x) = \max_{i \in \mathbb{m}} \left| f^{(i-1)}(x^+) - f^{(i-1)}(x^-) \right| < \infty \]
which is unique. Also, from the definition of the impulsive sets \( \text{IMP}_j(x) \).

Now, assume that \( x \in \{ \text{IMP}_{i+1} \cap \text{IMP}_{j+1} \} \). Thus,
\[ 0 < \left| f^{(i)}(x^+) - f^{(j)}(x^-) \right| < \infty \Rightarrow f^{(i)}(x^+) - f^{(j)}(x^-) = \infty \]
from the definition of the impulsive sets. Then, \( x \in \text{IMP}_j(x) \). The opposite logic implication
\[ j = j(x) = \max_{i \in \mathbb{m}} \left| f^{(i)}(x^+) - f^{(j)}(x^-) \right| = \infty \]
is proved. Then, it has been fully proved that \( x \in \text{IMP} \Rightarrow \)
\[ x \in \text{IMP}_j \Leftrightarrow \exists a unique j = j(x) = \max_{i \in \mathbb{m}} \left| f^{(i)}(x^+) - f^{(j)}(x^-) \right| < \infty \]
Now, establish again a contradiction by assuming that
\[ j = j(x) = \max_{i \in \mathbb{m}} \left| f^{(i)}(x^+) - f^{(j)}(x^-) \right| = \infty \]
\[ f^{(i)}(x^+) - f^{(j)}(x^-) = 0 < \infty ; \forall k \in \mathbb{m} \]
what contradicts \( x \notin \text{IMP} \). This proves that the unique \( j = j(x) \) implying and being implied by \( x \in \text{IMP}_j \) satisfies
\[ f^{(i)}(x^+) - f^{(j)}(x^-) > 0 \]
Using the necessary–higher order distributional derivatives, one gets that
\[ x \in \text{IMP} \Rightarrow f^{(m)}(x) = \frac{(-1)^{m-j}(m-j)!}{x^m} \left( f^{(i)}(x^+) - f^{(j)}(x^-) \right) \delta(0) \]

; with \( j \in \mathbb{m} - 1 \cup \{0\} \) being uniquely defined so that
\[ 0 < f^{(i)}(x^+) - f^{(j)}(x^-) < \infty \] and nowhere continuous first-derivative defined as \( \bar{f}^{(i)}(x) = f^{(i)}(x^+) \) \( x \notin \text{IMP} \).

The above formula is applicable if \( f \notin PC^m(\mathbb{R}^+, \mathbb{R}) \) but it is also applicable if \( f \in PC^m(\mathbb{R}^+, \mathbb{R}) \) yielding:
\[ f^{(m)}(x^+) = \bar{f}^{(m)}(x) \text{ if } x \notin \text{IMP} \]
\[ f^{(m)}(x) = f^{(m)}(x^-) \text{ if } x \in \text{IMP} \]
\[ f^{(m)}(x^+) = \bar{f}^{(m)}(x+) \]
\[ f^{(m)}(x^-) = f^{(m)}(x^-) \]
if \( j = j(x) < m - 1 \)

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1459

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for a unique \( j(x) \in \mathbb{N} \) from Assertion 1. Denote further sets related to impulses as follows:

\[
\text{IMP}(x) := \{ z \in \text{IMP} : z < x \} \; ; \; \text{IMP}^{+}(x) := \{ z \in \text{IMP} : z \leq x \}
\]

being indexed by two subsets of integers of the same corresponding cardinals defined by:

\[
I(x) := j(x) \text{ indexing the members } z_i \text{ of } \text{IMP}(x) \text{ in increasing order}
\]

and

\[
I^{+}(x) := (j(x) + 1) \text{ indexing the members } z_i \text{ of } \text{IMP}^{+}(x) \text{ in increasing order}
\]

The following result holds:

**Theorem 5.4.** The Caputo fractional derivative of \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) of order \( \mu \in \mathbb{R}_+ \) satisfying \( m-1 < \mu \leq m; m \in \mathbb{Z}_+ \) and all \( x \in \mathbb{R}_+ \) is after using distributional derivatives becomes in the most general case:

\[
D_0^\mu f(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt
\]

\[
= \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f(t)dt
\]

\[
+ \sum_{i \in I(x)} \frac{1}{\Gamma(m-j(x)_i)} \int_{x_{i-1}}^{x_{i+1}} (x-t)^{m-j(x)_i} f^{(j(x)_i)}(t)dt
\]

\[
= \frac{1}{\Gamma(m-\mu)} \int_0^x \int_{x_{i-1}}^{x_{i+1}} (x-t)^{m-\mu-1} \tilde{f}(t)dt dt
\]

\[
+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} \int_{x_{i-1}}^{x_{i+1}} (x-t)^{m-\mu-1} \tilde{f}(t)dt dt
\]

\[
+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} \sum_{j \in \mathbb{Z}_+} (j-1)! \frac{1}{(x-x_{i})^{m-j(x)_i}} \int_{x_{i-1}}^{x_{i+1}} (x-t)^{m-j(x)_i} f^{(j(x)_i)}(t)dt
\]

\[
\left( D_0^\mu f \right)^{(m)}(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt
\]

\[
= \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f(t)dt
\]

\[
+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (j-1)! \frac{1}{(x-x_{i})^{m-j(x)_i}} \int_{x_{i-1}}^{x_{i+1}} (x-t)^{m-j(x)_i} f^{(j(x)_i)}(t)dt
\]

\[
\left( D_0^\mu f \right)^{(m)}(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt
\]

\[
+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (j-1)! \frac{1}{(x-x_{i})^{m-j(x)_i}} \int_{x_{i-1}}^{x_{i+1}} (x-t)^{m-j(x)_i} f^{(j(x)_i)}(t)dt
\]

\[
\left( D_0^\mu f \right)^{(m)}(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt
\]

\[
+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (j-1)! \frac{1}{(x-x_{i})^{m-j(x)_i}} \int_{x_{i-1}}^{x_{i+1}} (x-t)^{m-j(x)_i} f^{(j(x)_i)}(t)dt
\]

\[
\left( D_0^\mu f \right)^{(m)}(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt
\]

\[
+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (j-1)! \frac{1}{(x-x_{i})^{m-j(x)_i}} \int_{x_{i-1}}^{x_{i+1}} (x-t)^{m-j(x)_i} f^{(j(x)_i)}(t)dt
\]

\[
\left( D_0^\mu f \right)^{(m)}(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt
\]

\[
+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (j-1)! \frac{1}{(x-x_{i})^{m-j(x)_i}} \int_{x_{i-1}}^{x_{i+1}} (x-t)^{m-j(x)_i} f^{(j(x)_i)}(t)dt
\]

\[
\left( D_0^\mu f \right)^{(m)}(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t)dt
\]