Bifurcation Method for Solving Positive Solutions to a Class of Semilinear Elliptic Equations & Stability Analysis of Solutions

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Abstract—Semilinear elliptic equations are ubiquitous in natural sciences. They give rise to a variety of important phenomena in quantum mechanics, nonlinear optics, astrophysics, etc because they have rich multiple solutions. But the nontrivial solutions of semilinear equations are hard to be solved for the lack of stabilities, such as Lane-Emden equation, Henon equation and Chandrasekhar equation. In this paper, bifurcation method is applied to solving semilinear elliptic equations which are with homogeneous Dirichlet boundary conditions in 2D. Using this method, nontrivial numerical solutions will be computed and visualized in many different domains (such as square, disk, annulus, dumbbell, etc).

Keywords—Semilinear elliptic equations; positive solutions; bifurcation method; isotropy subgroups.

I. INTRODUCTION

In this paper, we study semilinear elliptic boundary value problems of the form

\[
\begin{align*}
    \Delta u + f(x, u(x)) &= 0, & \text{in } \Omega, \\
    u &> 0, & \text{in } \Omega, \\
    u &= 0, & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded open domain in \( \mathbb{R}^2 \), and \( f \) is a nonlinear function of \( x \) and \( u \). We will deal with \( f \equiv u^p, \lambda_1 u + u^p, p > 1 \), which are elliptic equations (2), (3) below:

\[
\begin{align*}
    \Delta u + u^p &= 0, & \text{in } \Omega, \\
    u &> 0, & \text{in } \Omega, \\
    u &= 0, & \text{on } \partial \Omega, \\
\end{align*}
\]

\[
\begin{align*}
    \Delta u + \lambda_1 u + u^p &= 0, & \text{in } \Omega, \\
    u &> 0, & \text{in } \Omega, \\
    u &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

Since 60’s of the 20th century, the existence and multiplicity of solutions to the boundary value problems of the nonlinear elliptic PDEs such as problems (2), (3) have been studied by the monotone iterative method in the ordered Banach space, the mountain pass lemma and the min-max theorem in the critical point theory. It becomes an important field of study. But what distribution and structure the solutions have and how to compute them have attracted the attention of many mathematicians, physicists and engineers.

There are mainly five numerical methods for computing such kinds of problems: the Monotone Iterative Scheme (MIS)\(^{[5,6]}\), the Mountain Pass Algorithm (MPA)\(^{[7]}\), the High Linking Algorithm (HLA)\(^{[8]}\), the Min-Max Algorithm (MMA)\(^{[9,10]}\) and the Search Extension Method (SEM)\(^{[11]}\). MIS is based on the monotone iterative methods in the ordered Banach space; MPA, MMA and HLA are based on the numerical implement of the mountain pass lemma and the min-max theorem in the critical point theory. MPA was proposed by Choi and McKenna to compute the solutions with the Morse Index (MI) 0 or 1. Ding, Costa and Chen established HLA for sign-changing solution (MI=2) of semilinear elliptic problems. Li and Zhou designed a new min-max algorithm (MMA) to find multiple saddle points with any Morse index which is more constructive than the traditional min-max theorem. Chen and Xie proposed SEM, which searches the initial guess based on the linear combination of the eigenfunctions of the linearized problem and then gets the better initial guess by the continuation method for the discretized problem by the finite element method.

The advantages of the bifurcation method are computation of the solutions to problem (1) with any Morse index and different symmetries as many as possible and simplification of the computation of problem (1). On the other hand, the difficulty in searching the initial guess in other methods can be solved effectively by the bifurcation method. The bifurcation method is applied successfully to solving the BVP of the Henon equation\(^{[12,13]}\).

The organization of our paper is as follows. In Sec.2, we introduce our idea of bifurcation method and present some definitions, theorems which will be used in the following sections. In Sec.3, we use bifurcation method to compute nontrivial positive solution of equation (3) with \( \Omega = [0, 1] \times [0, 1] \) and analyze stability of this solution. In Sec.4, we compute and visualize nontrivial positive solutions of equation (3) on many different complex domains.

II. THEORY & BACKGROUND

Models of (1) arise naturally in physics, engineers, biology and ecology, etc. Although nonlinearities may appear in seemingly endless form, the simplest and most basic form of nonlinearity is the power type. If we set \( f \equiv u^p, p > 1 \), which...
is (2) we have mentioned, called Lane-Emden (-Fowler) equation and proposed by Chandrasekhar and Fowler\cite{14,15}. If we set \( f \equiv |x|^2 u^{\mu}, p > 1, j > 0 \), which is called Henon equation, studied by Henon\cite{16}. If we set \( f \equiv 4(2u + u^2)^{3/2} \), which is called Chandrasekhar equation, proposed by Chandrasekhar, Lieb and Yau\cite{14,17}. The equations above exhibit rich multiplicity of solutions, and draw many researchers interest, and there is a huge body of literature on them\cite{5-11}. Our focus here is positive solutions of equation (1).

We embed parameter \( \lambda \) in (1) and make the following form:

\[
\begin{align*}
\Delta u + \lambda u + f(x, u(x)) &= 0, \quad \text{in } \Omega, \\
u &> 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega, \\
\end{align*}
\]

where \( \lambda \in \mathbb{R} \). According to the bifurcation theory\cite{18,19}, Eq.4 has nontrivial solution branches bifurcated from the trivial solution near the bifurcation points. Along the nontrivial solution branches we can get the solutions to problems (1) by the continuation method when the parameter \( \lambda \) goes to 0.

In this paper we will illustrate bifurcation method by embedding (3) to the nonlinear bifurcation problems with parameter of the following form:

\[
\begin{align*}
\Delta u + \lambda u + u^p &= 0, \quad \text{in } \Omega, \\
u &> 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega. \\
\end{align*}
\]

Now, let’s start with some definitions and theorems which we will use to describe the followings.

**Definition 1.** For \( x \in \mathbb{R}^n \), a set \( \Sigma_x = \{ \gamma \in \Gamma \mid |x - x| = \varepsilon \} \) \( \Gamma \) is called isotropy subgroup of \( x \).

**Definition 2.** If \( d^i f = (i = 1, 2, \ldots, k - 1) \) is everywhere differentiable in \( U \), and \( d^{k+1} f : U \rightarrow \Psi(\mathbb{C}, \mathbb{C}^{k+1}, f(X, Y)) \) is differentiable at \( x_0 \in U \), then

\[
d^k f(x_0) = d(d^{k-1} f(x_0)) (x, X, X) = \mathcal{C}^k (X, Y)
\]

called \( k \)-th differential of \( f \) at \( x_0 \). And

\[
d^k f(x_0) = \left[ \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_k} f(x_0) + \sum_{i=1}^{k} \lambda_k \frac{\partial}{\partial \lambda_i} f(x_0) \right] |_{x_0 = \cdots = x_k = 0}.
\]

**Definition 3.** Assume \( f(x, \lambda, \lambda) = 0 \) is an equation with symmetry \( Z_2 \), \( f : X \times U \rightarrow \mathbb{R} \) is a Banach space, \( X' \) is a conjugate space of \( X \). Space \( X \) and \( X' \) can be decomposed into \( X = X_s \circ X_a, X' = X_s' \circ X_a' \), where

\[
X_s = \{ x \in X : S x = x \}, X_a = \{ x \in X : S x = -x \}, X_s' = \{ y \in X' : yS = y \}, X_a' = \{ y \in X' : yS = -y \}.
\]

If there exists a singular point \((x_0, \lambda_0)\) of \( f_2 \) on the solution branch \( C_2 = \{ (x, \lambda) : x \in X_s \} \), then

\[
N(f_2) = \text{span} \{ \varphi_0 \}, \quad \varphi_0 \in X_s, \\
R(f_2) = \{ y \in X : \psi_T y = 0 \}, \quad \psi_0 \in X_a,
\]

together with \( \psi_0 (f_{2x} \varphi_0 + f_{2 \lambda} \lambda_0) \neq 0 \), where \( \psi_0 \) is the unique solution to \( f_{2x} \varphi_0 + f_{2 \lambda} \lambda_0 = 0 \), then \((x_0, \lambda_0)\) is called a pitchfork bifurcation point of \( f(x, \lambda) = 0 \) with symmetry \( Z_2 \).

Furthermore, if we assume:

\[
\psi_0 \varphi_0 \neq 0, \\
\psi_0 (f_{2x} \varphi_0 + f_{2 \lambda} \lambda_0) \neq 0,
\]

where \( \varphi_1 \in X_a \) is the unique solution to \( f_{0x} \varphi_0 + f_{0 \lambda} \varphi_1 = 0 \), then \((x_0, \lambda_0)\) is called simple third-order pitchfork bifurcation point of \( f(x, \lambda) = 0 \) with symmetry \( Z_2 \).

**Theorem 1.** (Implicit Function Theorem)

(1) Assume \( f(x, \lambda, \lambda) = 0, x \in X, \lambda_0 \in \mathbb{R} \), where \( f : X \times \mathbb{R} \rightarrow \mathbb{R} \), is continuous differentiable on their open domain,

\[
\| f(\lambda, \lambda) \| \leq M_0.
\]

Then there exists \( \lambda_1 > \lambda_0, \lambda_2 > \lambda_0 \), and for all \( \lambda \in (\lambda_0 - \lambda_1, \lambda_2 - \lambda_2) \), there exists \( x(\lambda) \in B_{\rho_0}(\lambda_0), B_{\rho_0}(\lambda_0) = \{ x \in X : \| x - x_0 \| < \rho_0 \} \), in which \( x = x(\lambda) \) with the following properties:

(4) \( x(\lambda) = x_0 \),

(5) \( f(x, \lambda, \lambda) = 0 \),

(6) \( f(\lambda, \lambda) = 0 \),

(7) \( x(\lambda) \) is continuous for \( \lambda \in (\lambda_0 - \rho_2, \lambda_0 + \rho_2) \).

**Proof.** C.f.[18].

**Theorem 2.** (Newton-Kantorovich)

Assume \( f : X \rightarrow X, X \) is a Banach space, and

\[
(1) \exists \varepsilon > X, f(x, \lambda) = 0, \quad \| f(\lambda, \lambda) \| \leq \beta.
\]

(2) \( f(\lambda, \lambda) \) is continuously differentiable on \( \Gamma \),

\[
(3) \exists \varepsilon > 0, \forall x, y \in X, \exists \lambda_0 \in (\lambda_0 - \rho_0, \lambda_0 + \rho_0), \| f(x - f(\lambda, \lambda)) \| \leq \gamma(\| x - y \|).
\]

Then series \( x_n \) in Newton iteration

\[
x_{n+1} = x_n - f(x(\lambda), \lambda) = 0,
\]

such that

(5) \( x_n \in B_{\rho_0}(\lambda) \),

(6) \( x_n \) convergence to \( x(\lambda) \) in \( B_{\rho_0}(\lambda) \), which is a unique root of \( f(x) = 0 \) when \( x \in B_{\rho_0}(\lambda) \).

**Proof.** C.f.[20].

**Theorem 3.** (Keller Lemma)

Assume \( A : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N \times \mathbb{R} \), and

\[
A = \begin{pmatrix} A & B \\ C^T & D \end{pmatrix},
\]

where \( A : \mathbb{R}^N \rightarrow \mathbb{R}^N, B : \mathbb{R}^L \rightarrow \mathbb{R}^N, C^T : \mathbb{R}^N \rightarrow \mathbb{R}^L, D : \mathbb{R}^L \rightarrow \mathbb{R}^N \), then

(1) If \( A \) is nonsingular, then \( A \) is nonsingular iff \( D - C^TA^{-1}B \) is nonsingular.

(2) If \( A \) is singular and \( \dim N(A) = l \), then \( A \) is nonsingular iff

\[
(\alpha) \quad \text{dim} R(B) = l, \quad (\beta) \quad R(B) \cap R(A) = \{0\},
\]

(\( c) \quad \text{dim} R(C^T) = l, \quad (d) \quad N(A) \cap \text{N}R(C^T) = \{0\}.
\]

(2) If \( A \) is singular and \( \dim N(A) > l \), then \( A \) is singular.

**Proof.** C.f.[21,18].

**Theorem 4.**

The nontrivial solution branch \( (x(\varepsilon), \lambda(\varepsilon)) = (x(\lambda(\varepsilon)) + \varepsilon(\Phi_0 + \varepsilon(\lambda(\varepsilon)) \lambda(\varepsilon)) \) bifurcate from symmetry solution branch \( (x(\lambda), \lambda) \) near the symmetry-breaking point \((x_0, \lambda_0)\) of simple
third-order pitchfork bifurcation, then the following assertions are valid.

\[ \lambda(0) = \lambda_0, \quad \omega(0) = 0, \]
\[ \lambda(\epsilon) = \lambda(\epsilon), \quad \lambda(\epsilon) = 0, \]
\[ \lambda(\epsilon) = -\frac{\psi_T^T f_{yy}^0 \varphi_0 \varphi_0 + 3 f_{yy}^0 \varphi_1 \varphi_0}{3 f_y^0 (f_{yy}^0 \varphi_0 + f_{y\lambda}^0 \varphi_0)}, \quad \omega(\epsilon) = \varphi_1/2. \]

**Proof.** C.f.[22,18].

III. UNIT SQUARE

(1) Analysis Assume \( \Omega = \Omega_0 = [0, 1] \times [0, 1] \), then Eq.5 turns into

\[ F(u, \lambda) = \left\{ \begin{array}{ll}
\Delta u + \lambda u + u^3 = 0, & (x, y) \in \Omega_0, \\
u > 0, & (x, y) \in \Omega, \\
u < 0, & (x, y) \in \partial \Omega. \end{array} \right. \quad (7) \]

Let \( D_4 = \{ I, R_1, R_2, S_1, S_2, S'_1, S'_2 \} \), where

\[ Iu(x, y) = u(x, y), \quad S_1 u(x, y) = u(x, 1 - y), \]
\[ S'_1 u(x, y) = u(1 - x, y), \quad S_2 u(x, y) = u(1 - y, x), \]
\[ R_1 u(x, y) = u(1 - y, x), \quad R_2 u(x, y) = u(y, 1 - x). \]

The problem (7) is \( D_4 \)-equivariant. Especially, if \( p \) is odd in (7), Eq.7 is \( \Gamma \)-equivariant, where \( \Gamma = D_4 \times Z_2 \), \( Z_2 = \{ 1, -1 \} \), namely \( F(\gamma u, \tau) = \gamma F(u, \lambda), \forall \gamma \in \Gamma \). The isotropy subgroups of \( D_4 \) are \( D_4 = \{ I, R_1, R_2, R_3, S_1, S_2, S'_1, S'_2 \} \), \( Z_2 = \{ 1, -1 \} \), \( \Sigma_1 = \{ I, R_1, R_2, R_3 \}, \quad \Sigma_2 = \{ I, R_2 \}, \Sigma_1 = \{ S_1, S'_1 \}, \quad \Sigma_2 = \{ S_2, S'_2 \} \), \( \Sigma_1 = \{ I, R_2, S_2, S'_2 \}, \Sigma_2 = \{ I, R_1, S_1, S'_1 \} \). Let \( \Sigma \) be one of the above isotropy groups and \( X^\Sigma \) be the invariant subspace of \( \Sigma \), then the equation (7) yields

\[ F_\Sigma(u, \lambda) = 0. \quad (u, \lambda) \in X^\Sigma \times \mathbb{R} \]  

(8)

Consider the linearized equation of (7) at \( u = 0 \), we get

\[ \left\{ \begin{array}{ll}
\Delta \varphi + \lambda \varphi = 0, & (x, y) \in \Omega, \\
\varphi > 0, & (x, y) \in \Omega_0, \\
\varphi = 0, & (x, y) \in \partial \Omega. \end{array} \right. \quad (9) \]

It is well known that Eq.9 always has a trivial solution if we don’t consider \( \varphi > 0 \). Further more, Eq.9 has eigenvalues \( \lambda_{n,m} = (n^2 + m^2)\pi^2 \) and corresponding eigenfunctions \( \varphi_{n,m} = \sin(n\pi x)\sin(m\pi y) \). Therefore, \( \varphi_{n,m} = \left( \sum \varphi_{n,m} \right) \) are roots of Eq.9 when \( \lambda = \lambda_{n,m} = (n^2 + m^2)\pi^2 \). From theory of symmetry-breaking, we know that \( \lambda_{n,m} = (n^2 + m^2)\pi^2 \), \( n, m = 1, 2, \cdots \) are bifurcation points of (9), and there are nontrivial solutions with different symmetries bifurcate from these bifurcate points (see Table 1).

From the analysis above, we know that the solution branch which bifurcates from the first bifurcation point \( 2\pi^2 \) is a positive solution branch. Bifurcation method will be applied to compute the positive solution of (7), and stability analysis of this solution is in the subsequent pages.

(2) Algorithm For \( \lambda_0 = \lambda_{1,1} = 2\pi^2, \varphi_{1,1} = \sin(\pi x)\sin(\pi y) \), let

\[ L = \Delta + 2\pi^2, \quad X = \{ u \mid u \in C^2(\Omega), u|_{\partial \Omega_0} = 0 \}, \]
\[ Y = \{ u \mid u \in C^0(\Omega) \}. \]

we define inner product by \( \langle u, v \rangle = 4 \int_0^1 \int_0^1 u(x, y) \partial x \partial y \), \( L \) is a Fredholm self-adjoint operator with index zero, and

\[ N(L^*) = N(L) = \text{span}\{\varphi_{1,1}\} := \text{span}\{\varphi_0\}, \quad (10) \]

where \( N(L) \) and \( N(L^*) \) are the null space of \( L \) and \( L^* \) respectively. Space \( X \) and \( Y \) have the decomposition

\[ X = N(L) \oplus M, \quad Y = N(L) \oplus R(L), \]

where \( M = N(L)^\perp \cap X, \quad R(L) \) is the range of \( L \).

Let \( P \) be the orthogonal projector from \( Y \) to \( R(L) \)

\[ Pz = z - \langle z, \varphi_0 \rangle \varphi_0. \quad z \in Y \]

Eq.7 is equivalent to

\[ PF(\tau \varphi_0 + \omega, \mu, \lambda_0) = 0, \quad \tau \in \mathbb{R}, \omega \in M \]

(11)

\[ \langle \varphi_0, F(\tau \varphi_0 + \omega, \mu, \lambda_0) \rangle = 0. \]

(12)

where \( \mu = \lambda - \lambda_0, \omega = \tau \varphi_0 + \omega. \) Since \( PF(0, \lambda_0) = PF_0(0, \lambda_0) = PL = L \), and \( L \) restricted in \( M \) is regular, Eq.11 has a unique solution \( \omega = \omega(\tau, \mu) \) which satisfies \( \omega(0, 0) = 0 \) by Theorem 1.

Substituting \( \omega(\tau, \mu) \) into (12) yields

\[ g(\tau, \mu) = \langle \varphi_0, F(\tau \varphi_0 + \omega(\tau, \mu), \mu) \rangle = 0. \]

(13)

Then we get

\[ F(u, \lambda) = F(\tau \varphi_0 + \omega, \mu + \lambda_0) = \Delta \omega + \lambda_0 \omega + h(\tau, \mu), \quad (14) \]

where \( h(\tau, \mu) = \mu(\tau \varphi_0 + \omega) + \langle \varphi_0, \lambda \rangle \omega = \omega(\tau, \mu). \) From Definition 2 above, we get

\[ (d^k F(u, \lambda_0)(\nu_1, \cdots, \nu_k) = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} F(\sum_{i=1}^k t_i \nu_i, \lambda)|_{t_1=\cdots=t_k=0} \]

\[ = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} \left( \Delta \sum_{i=1}^k t_i \nu_i + \lambda_0 (\sum_{i=1}^k t_i \nu_i) + (\sum_{i=1}^k t_i \nu_i)^2 \right)|_{t_1=\cdots=t_k=0} \]

\[ = \left\{ \begin{array}{ll}
0, & k < p, \\
\sum_{i=1}^p \sum_{j=1}^k \nu_i, & k \geq 2. 
\end{array} \right. \]

(15)
Especially

\[(d^kF)(\varphi_0, \varphi_1, \ldots, \varphi_n) = \begin{cases} 0, & k \neq p \\ p!L_p^2, & k = p. \end{cases} \quad (16)\]

Differentiating Eq.11 with respect to \(\tau\), we get

\[PdF(\varphi_0 + \omega_\tau) = 0, \quad (17)\]

which is evaluated at \((0, 0)\) leads to \(L\omega_\tau(0, 0) = 0\) due to \(dF(0, \lambda_0) = L, \varphi_0 \in N(L), P_L = L\). Since \(\omega_\tau(0, 0) \in M\) and \(L\) restricted in \(M\) is regular, it follows that

\[\omega_\tau(0, 0) = 0. \quad (18)\]

Similarly, differentiating Eq.12 with respect to \(\tau\), we get

\[g_\tau(\tau, \mu) = (\varphi_0, dF(\varphi_0 + \omega_\tau)), \quad (19)\]

therefore

\[g_\tau(0, 0) = (\varphi_0, dF(\varphi_0)) = 0. \quad (20)\]

Similarly, from (15) we get

\[
\begin{align*}
\omega_\tau(0, 0) &= \left\{ \begin{array}{ll}
-2L^{-1}P_{\varphi_0}^2, & \text{if } p = 2, \\
0, & \text{if } p \geq 3.
\end{array} \right. \\
g_\tau(0, 0) &= \left\{ \begin{array}{ll}
\frac{12\omega_{2, \mu_2}}{\tau}, & \text{if } p = 2, \\
0, & \text{if } p \geq 3.
\end{array} \right. \\
\omega_\tau(0, 0) &= \left\{ \begin{array}{ll}
-6L^{-1}(P_L\varphi_0), & \text{if } p = 2, \\
0, & \text{if } p \geq 3.
\end{array} \right. \\
g_\tau(0, 0) &= \left\{ \begin{array}{ll}
\frac{12}{\pi}(\varphi_0)^2L^{-1}P_{\varphi_0}^2, & \text{if } p = 2, \\
0, & \text{if } p \geq 3.
\end{array} \right. \\
\end{align*}
\]

Therefore we have approximately

\[
\begin{align*}
\omega_\tau(\mu, \mu) &= \left\{ \begin{array}{ll}
\frac{3}{\pi} \sin(3(x_\mu)), & \text{if } p = 2, \\
0, & \text{if } p \geq 3.
\end{array} \right. \\
g_\tau(\mu, \mu) &= \left\{ \begin{array}{ll}
\frac{1}{\pi x_\mu}\sin\left(\frac{3}{\pi}x_\mu\right), & \text{if } p = 2, \\
0, & \text{if } p \geq 3,
\end{array} \right. \\
\end{align*}
\]

Next we want to get the approximative analytic solution of (7). Here we deal with Eq.7 while \(p = 3, \lambda_1 = 1\). Substituting \(\mu = \lambda_1 - \lambda_0\) into (41), we can get

\[
\tau = \frac{4}{3}\sqrt{\tau - \mu} = \frac{4}{3}\sqrt{\lambda_1 - \lambda_0} = \frac{4}{3}\sqrt{2\pi^2 - 1}.
\]

Then we have

\[
u = \frac{4}{3}\sqrt{2\pi^2 - 1} \times \frac{3}{3}\sqrt{2\pi^2 - 1}, 1 - 2\pi^2).
\]

In order to know \(\omega_\tau(\sqrt{2\pi^2 - 1}, 1 - 2\pi^2)\) in (42), we get \(\omega_\tau(\tau, \mu) = \frac{1}{2}\omega_{2, \mu}(0, 0)\) from (40). When \(p = 3\), differentiating Eq.11 with respect to \(\tau\) three times, then we get

\[6P_{\varphi_0}^3 + L\omega_\tau(0, 0) = 0. \quad (43)\]

Due to

\[
\begin{align*}
\varphi_0 &= \sin^3(\pi x)\sin^3(\pi y) = \frac{9}{16}\sin(\pi x)\sin(\pi y) - \frac{3}{16}\sin(3\pi x)\sin(3\pi y) \\
&\quad \quad \quad - \frac{3}{16}\sin(3\pi x)\sin(3\pi y),
\end{align*}
\]

we get \(P_{\varphi_0} = P\sin^3(\pi x)\sin^3(\pi y)\), together with \(\omega_{2, \mu}\) is restricted in \(X_\mu\), then we can let \(\omega_{2, \mu}(0, 0) = C\sin(3\pi x)\sin(3\pi y)\), where \(C\) is a undetermined constant. Substituting \(\omega_{2, \mu}(0, 0)\) into (43) yields

\[
\omega_{2, \mu}(0, 0) = \frac{3}{128\pi^2}\sin(3\pi x)\sin(3\pi y). \quad (45)
\]

From (42) we can get the approximative positive analytic solution of Eq.7

\[u = \frac{4}{3}\sqrt{2\pi^2 - 1}\sin(\pi x)\sin(\pi y) + \left(\frac{2\pi^2 - 1}{128\pi^2}\right)\sin(3\pi x)\sin(3\pi y). \quad (46)
\]

(3). Stability analysis

We always have trivial solution branch \((u, \lambda) = (0, \lambda)\) for equation \(F(u, \lambda) = u + \lambda u + u^3 = 0\). When the trivial solution branch cross the bifurcation point \((0, \lambda_0)\), one eigenvalue of \(F_u^0\) equals zero, and all the others are less than 0. So we can know that the sign of the "special" eigenvalue determines stability of the trivial solution when eigenvalue \(\lambda\) cross \(\lambda_0\).

As mentioned in Sec.2, the meaning of \(\varphi_0, \varphi_0, \varphi_1, \varphi_2\) are given by Definition 3. In addition, we construct \(l_0\) which satisfies \(l_0\varphi_0 = 1\).
branch \((u, \lambda) = (\bar{u}(\varepsilon), \lambda(\varepsilon))\), let
\[
H(z, \varepsilon) = \left( F_u(\bar{u}(\varepsilon), \lambda(\varepsilon)) \varphi(\varepsilon) - \sigma_1 \varphi(\varepsilon) \right) = 0,
\]
where \(z = (\varphi, \sigma_1), z_0 = (\varphi_0, 0)\). \(H^0_z = \left( \frac{F^0_u}{t^0_{\varphi^0}} - \frac{\varphi^0}{t^0_{\varphi}} \right)\) is
nonsingular at \(\varepsilon = 0\). So we get that Eq\.49 has a unique solution branch \((z(\varepsilon), \varepsilon) = (\varphi(\varepsilon), \sigma_1(\varepsilon), \varepsilon)\) which satisfies
\(\sigma_1(\lambda_0) = 0, \varphi(\lambda_0) = \varphi_0\) by the Implicit function theorem. Differentiating \(F_u(\bar{u}(\varepsilon), \lambda(\varepsilon))\varphi(\varepsilon) - \sigma_1 \varphi(\varepsilon) = 0\) with respect to \(\varepsilon\) at \(\varepsilon = 0\),
we get
\[
|F^0_u u_{\lambda_0} \varphi(0) + F^0_u \varphi(0) \lambda_0 - \sigma'(0) \varphi(0)| = 0.
\]
we choose $N = 200$, $h = 1/200$.

**Step 2:** We store the data of the five-point difference operator $\Delta_h$ in $A$, and compute the eigenvalue problem $A\varphi_h = \lambda_h \varphi_h$, where $\varphi_h$ and $\lambda_h$ are approximations to $\varphi_0$ and $\lambda_0$.

**Step 3:** Let

$$u = \tau \varphi_h + \omega, \quad \eta = \lambda - \lambda_h,$$

where $\tau$ is a small parameter, and $\omega$ satisfies $(\varphi_h, \omega) = 0$. Substituting (53) into (5), we have

$$\begin{align*}
\Delta \omega + (\eta + \lambda_h)\omega + \eta \tau \varphi_h + \omega_p &= 0, \quad (x, y) \in \Omega_0, \\
\omega &= 0, \quad (x, y) \in \Omega_0, \\
(\varphi_h, \omega) &= 0.
\end{align*}$$

The Gauss-Newton method is used to solve this nonlinear equation for different $\tau$ from $\tau = 0$ to end $\tau = \tau_{end}$; $\tau_{end}$ must be chosen big enough in order that the nontrivial solutions of (54) are faraway from the trivial solution.

**Step 4:** Continue $\lambda$ until $\lambda = \lambda_1$, and then we get the nontrivial solution $u(x, y)$ of (3) and plot it.

(2) **Visualization of positive solutions of (3) in many complex domains** ($p = 3, \lambda_1 = 1$)

<table>
<thead>
<tr>
<th>Shape of the domain</th>
<th>Bifurcation point</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit square (Fig.3)</td>
<td>19.739</td>
<td>$D_4$</td>
</tr>
<tr>
<td>Unit disk (Fig.4)</td>
<td>22.976</td>
<td>$O(2)$</td>
</tr>
<tr>
<td>L-shaped domain (Fig.5)</td>
<td>38.576</td>
<td>$S_2^1$</td>
</tr>
<tr>
<td>Unsymmetrical annulus (Fig.6)</td>
<td>49.102</td>
<td>$S_2^1$</td>
</tr>
<tr>
<td>Annulus (Fig.7)</td>
<td>212.166</td>
<td>$O(2)$</td>
</tr>
<tr>
<td>The exterior of a “Butterfly” (Fig.8)</td>
<td>64.805</td>
<td>$S_2^d$</td>
</tr>
<tr>
<td>Heart (Fig.9)</td>
<td>60.555</td>
<td>$S_2^1$</td>
</tr>
<tr>
<td>Crisscross (Fig.10)</td>
<td>57.610</td>
<td>$D_4$</td>
</tr>
<tr>
<td>Ellipse (Fig.11)</td>
<td>56.382</td>
<td>$S^M_2$</td>
</tr>
<tr>
<td>Dumbbell shaped domain (Fig.12)</td>
<td>189.157</td>
<td>$S^M_2$</td>
</tr>
</tbody>
</table>

**TABLE II**

THE SOLUTIONS WITH DIFFERENT SYMMETRIES TO EQ.3

($p = 3, \lambda_1 = 1$)
ACKNOWLEDGMENT

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