The Structure of Weakly Left C-wrpp Semigroups

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Abstract—In this paper, the class of weakly left C-wrpp semigroups which includes the class of weakly left C-rpp semigroups as a subclass is introduced. To particularly show that the spined product of a left C-wrpp semigroup and a right normal band which is a weakly left C-wrpp semigroup by virtue of left C-full Ehresmann cyber groups recently obtained by authors Li-Shum, results obtained by Tang and Du-Shum are extended and strengthened.

Keywords—Left C-semigroup, left C-wrpp semigroup, left quasi-normal band, weakly left C-wrpp semigroup

I. INTRODUCTION

Throughout this paper, we adopt the notation and terminologies given by Howei[1] and Li-Shum[2].

By modifying Green’s relations on rpp semigroups, Tang [3] has introduced a new set of Green’s relations on a semigroup $S$ and by using these new Green’s relations, he was able to give a description for a wider class of C-rpp semigroups, namely, the class of C-wrpp semigroups. Tang [3] considered a Green-like right congruence relation $L^a$ on a semigroup $S$ for $a,b \in S$ $aL^ab$ if and only if $aRx ay \Leftrightarrow bxR by$ for all $x,y \in S^1$. Moreover, Tang pointed out in [3] that a semigroup $S$ is a wrpp semigroup if and only if each $L^a$-class of $S$ contains at least one idempotent.

Recall that a wrpp semigroup $S$ is a C-wrpp semigroup if the idempotents of $S$ are central. It is well known that a semigroup $S$ is a C-wrpp semigroup if and only if $S$ is a strong semilattice of left cancellative monoids (see [3]). Because a Clifford semigroup can be expressed as a strong semilattice of groups and a C-rpp semigroup can be expressed as a strong semilattice of left cancellative monoids (see [4]), we see immediately that the concept of C-wrpp semigroups is a common generalization of Clifford semigroups and C-rpp semigroups.

For wrpp semigroups, Du-Shum [5] first introduced the concept of left C-wrpp semigroups, that is, a left C-wrpp semigroup whose satisfy the following conditions: (i) for all $e \in E(L^a)$, $a = ae$ where $E(L^a)$ is the set of idempotents in $L^a$; (ii) for all $a \in S$, there exists a unique idempotent $a^+$ satisfying $aL^a a^+$ and $a = a^+ a$; (iii) for all $a \in S$, $aS \subseteq L^a(a)$, where $L^a(a)$ is the smallest left $a$-ideal of $S$ generated by $a$.


Guo [6] has investigated weakly left C-semigroups, and he pointed out that a semigroup $S$ is a weakly left C-semigroup if and only if $S$ is a completely regular semigroup with idempotents set $E(S)$ forming a left quasi-normal band.

In this paper, we first define the concept of weakly left C-wrpp semigroups. A structure theorem for weakly C-wrpp semigroups is obtained, and we prove this theorem in view of the structure of left C-full Ehresmann cyber groups recently obtained by Li and Shum [2].

II. PRELIMINARIES

We first recall that some known results used in the sequel.

The following results due to [2] and [7].

Let $S$ be a semigroup, and $U$ a subset of the set $E(S)$ which is the set of all idempotents of $S$. For all $a \in S$, let $U^a = \{u \in U | au = a\}$, $U^a^* = \{u \in U | ua = a\}$, $U_a = U^a \cup U^a^*$, $U_a = \{u \in U | ua = a = au\}$. According to Lawson [8] and He [7], we have the following relations on $S$:

$L^U = \{(a,b) \in S \times S | U^a = U^a^*\}$,
$R^U = \{(a,b) \in S \times S | U^a = U^a^*\}$,
$H^U = L^U \cup R^U$,
$Q^U = \{(a,b) \in S \times S | U_a = U_a^*\}$.

It is easy to verify that above relations are equivalent relations. For all $a \in S$, a $L^U$-class, a $R^U$-class, a $H^U$-class and a $Q^U$-class of $S$ containing $a$, are denoted by $L^U_a$, $R^U_a$, $H^U_a$ and $Q^U_a$, respectively. For the sake of convenience, we denote the semigroup $S$ with a projective set $U$ which is a subset of all idempotents $E(S)$ by $S(U)$.

Consider the special semigroup $S(U)$ with $U = E(S)$.

Then the equivalent relations on $S = S(U)$, say $L^{E(S)}$, $R^{E(S)}$, $H^{E(S)}$ and $Q^{E(S)}$, respectively. For brevity, we write $L$, $R$, $H$ and $Q$, respectively.

Definition 2.1 A semigroup $S(U)$ is called a $U$-semi-lpp semigroup if each $R^U$-class of $S$ contains at least one element in $U$, that is, $R^U_a \cap U \neq \emptyset$ for all $a \in S$. A semigroup $S(U)$ is called a $U$-semi-rpp semigroup if each $L^U$-class of $S$ contains at least one element in $U$, that is, $L^U_a \cap U \neq \emptyset$ for all $a \in S$.
A semigroup \( S(U) \) is called a \( U \)-semiabundant semigroup if both each \( L^U \)-class and each \( R^U \)-class of \( S \) contain at least one element in \( U \), that is, \( \vec{L}_U \cap U \neq \emptyset \) and \( \vec{R}_U \cap U \neq \emptyset \) for all \( a \in S \). A semigroup \( S(U) \) is called \( U \)-abundant semigroup if each \( \vec{L}_U \)-class of \( S \) contains at least one element in \( U \), that is, \( \vec{L}_U \cap U \neq \emptyset \). Denoted the unique element in \( U \) by \( a' \) if \( |\vec{L}_U \cap U| = 1 \). In special case \( U = E(S) \), a \( E(S) \)-abundant semigroup is called a full abundant semigroup, in this case, \( a' \) is usually written as \( a' \).

Lawson\[^8\] point out that \( \Gamma^U \) is not necessarily a right congruence and \( R^U \) is not necessarily a left congruence on \( S(U) \). We have

**Definition 2.2** A semigroup \( S(U) \) is called satisfying \( U \)-right(left) congruence condition if \( L^U \subset RC(S) \cup \{a' \} \cup \{a'' \} \) and \( R^U \subset LC(S) \cup \{a' \} \cup \{a'' \} \), a semigroup \( S(U) \) is called satisfying \( U \)-congruence condition if \( L^U \subset RC(S) \) and \( R^U \subset LC(S) \), where RC(S) is a lattice that all right congruences form, and LC(S) is a lattice that all left congruences form.

**Definition 2.3** A \( U \)-semiabundant semigroup \( S(U) \) is called an Ehresmann semigroup if \( S(U) \) satisfies \( U \)-congruence condition and \( U \) is a subsemilattice of \( U \). In particular, a Ehresmann semigroup \( S(U) \) is a C-Ehresmann semigroup if \( U \) lies in center of \( S(U) \).

**Definition 2.4** A \( U \)-abundant semigroup \( S(U) \) is called an orthodox \( U \)-abundant semigroup if \( U \) is a subsemilattice of \( S(U) \) and \( (ab)_U \equiv (a'b')_U \) for all \( a, b \in S \), where \( D = L \lor R \) is a usual Green’s \( D \) relation.

**Definition 2.5** An orthodox \( U \)-abundant semigroup is called a \( C \)-Ehresmann semigroup if \( aS \subseteq S_{a} \) holds for all \( u \in U \).

**Definition 2.6** An orthodox \( U \)-abundant semigroup is called a left \( C \)-Ehresmann semigroup if the identity \( uxy = uxy \) holds for all \( u \in U, x, y \in S \).

When \( U = E(S) \), we call an orthodox \( E(S) \)-abundant semigroup\(left \ C \)-Ehresmann semigroup, left \( C \)-Ehresmann [orthodox full \( C \)-Ehresmann semigroup, left \( C \)-Ehresmann [orthodox full \( C \)-Ehresmann semigroup, left \( C \)-Ehresmann cyner group].

Recall that the direct product \( I \times T \) of a left zero band \( I \) and a monoid \( T \) is called a left monoid, and the direct product \( I \times T \) of a left zero band \( I \) and an unipotent semigroup \( T \) is called a left C-unipotent semigroup. It is well known that a right normal band \( \Lambda \) can be expressed as a strong semilattice of right zero bands, that is, \( \Lambda = [Y; A_u; \varphi_u, \beta] \).

By using the above results, \( \Lambda \) have proved that the following results:

**Lemma 2.7** The following statements are equivalent for a semigroup \( S \):

(i) \( S(U) \) is a left \( C \)-Ehresmann semigroup for some \( U \subseteq E(S) \);

(ii) \( S \) is a semilattice \( Y \) of left monoids \( S_a = I_a \times T_a \) and \( U = \{(i, I_a) \mid i \in I_a \} \) is a subsemigroup of \( S \) where \( a \in Y, I_a \) is the identical element in \( T_a \).

By virtue of above lemma, a left \( C \)-Ehresmann semigroup \( S(U) = [Y; S_a = I_a \times T_a] \) may be defined as a semilattice of left monoids \( S_a = I_a \times T_a \) and the set \( U = \{(i, I_a) \mid i \in I_a, a \in Y \} \) is also a subsemigroup of \( S(U) \).

The following lemma have been recently proved by Li-Shum\[^2\].

**Lemma 2.8** Let \( S \) be a semigroup. Then \( S(U) \) is a left \( C \)-Ehresmann cyner group for some \( U \subseteq E(S) \) if only if \( S \) is isomorphic to a spined product \( S_a \times \Lambda \) of a left \( C \)-Ehresmann semigroup \( S_a = [Y; I_a \times T_a] \) and a right normal band \( \Lambda = [Y; A_u; \varphi_u, \beta] \) with respect to the semilattice \( Y \) (see [2]).

III. THE STRUCTURE OF WEAKLY LEFT \( C \)-WRPP SEMIGROUPS

In this section, the concept of weakly left \( C \)-wrpp semigroups is introduced. We shall prove that a structure theorem for weakly left \( C \)-wrpp semigroups. First, we introduce the concept of weakly left \( C \)-wrpp semigroups.

**Definition 3.1** A semigroup \( S \) is called a weakly left \( C \)-wrpp semigroup if \( S \) is a strong wrpp semigroup and satisfy identity \( exy = eyx \) for all \( e \in E(S) \) and \( x, y \in S \).

According to [5], we know that the left \( C \)-wrpp semigroup is a special case of the weakly left \( C \)-wrpp semigroup.

**Lemma 3.2** Let \( S \) be a strongly wrpp semigroup. Then \( S \) is a full abundant semigroup with \( a^* = a^* \), for all \( a \in S \).

**Proof.** Let \( S \) be a strongly wrpp. To prove \( S \) is a full abundant semigroup, we only need to prove \( a^* = a^* \) for all \( a \in S \). Let \( I_a = \{e \mid ea = ae = a\} \). For all \( e \in I_a \), since \( (a'e) = (a'e) = (a'e)a = a(a'e)(a'e) = a^* = a^* \), and \( a = aL^* \lor a = a = aL^* \lor a = a \). So we have \( (a'e) = a^* \). This implies that \( (a'e) = a^* \). Thus, we have \( a^* = a^* = a' \) and whence \( a^* = a' \) in \( E(S) \). Consequently, we obtain that \( a = a \). On the other hand, we can easily verify that \( ea^* = a^* \). Therefore, \( a = a^* \), and so \( e \in I_a \). Thus, it follows that \( I_a \subseteq I_a \).

Clearly, \( I_a \subseteq I_a \) and whence \( I_a = I_a \). This means that \( (a,e^*) \in Q \). Therefore, \( S \) is a full abundant semigroup with \( a^* = a^* \). The proof is completed.

**Lemma 3.3** Let \( S \) be a weakly left \( C \)-wrpp semigroup. Then
This means that $(a\ b)^2$ is isomorphic to $a\ b = a\ b$. Since $(a\ b)^2, (a\ b)^3 \in \mathcal{E}(S)$, we can verify that $(a\ b)^2 \mathcal{L}_o (a\ b)^3$, and hence $(a\ b)^2 = (a\ b)^3$. For all $x, y \in S$, we have $(a\ b)^2 x (a\ b)^3 y \Rightarrow (a\ b)^2 x (a\ b)^3 y \Rightarrow (a\ b)^2 x (a\ b)^3 y$.

We now characterize the weakly left C-wrpp semigroups. Theorem 3.4 A semigroup $S$ is a weakly left C-wrpp semigroup if and only if $S$ is isomorphic to a spined product $S_{1} \times \Lambda$ of a left C-wrpp semigroup $S_{1} = [Y; S_{1} = I_{x} \times T_{x}]$ and a right normal band $\Lambda = [Y; \Lambda_{y}; \varphi_{x, y}]$ with respect to semilattice $Y$.

Proof. Necessity. Let $S$ be a weakly left C-wrpp semigroup. Then by Lemma 3.3, $S$ is a left C-full Ehresmann semigroup with $a' = a^*$ for all $a \in S$. According to Lemma 2.8, we know that $S$ can express as $S_{1} \times \Lambda$, where $S_{1} = [Y; S_{1} = I_{x} \times T_{x}]$ is a left C-full Ehresmann semigroup and $\Lambda = [Y; \Lambda_{y}; \varphi_{x, y}]$ is a right normal band. We only need to show that $S_{1}$ is a strongly wrpp semigroup and $T_{x}$ is a left C-rpp cancellative monoid.

For all $(i, a) \in S_{a} \times (j, b) \in S_{b}$ and $(k, c) \in S_{c}$, we have $(i, a)(j, b)(k, c) \Rightarrow (i, a)(i, 1_{a})(j, b)(i, 1_{a})(i, 1_{a})(k, c)$.

This means that $((i, a), (j, b), (k, c)) \Rightarrow (i, a), (i, 1_{a}), (j, b), (i, 1_{a}), (k, c)$, and hence $(a, b, c) \in T_{x}$ such that $ab = ac$, then $(i, a)(j, b)(i, 1_{a})(c) \Rightarrow (i, a)(j, b)(i, 1_{a})(c)$, and whence $bc \in T_{x}$, which means that $T_{x}$ is a left C-rpp cancellative monoid.

Sufficiency. Assume that $S = S_{1} \times \Lambda$ where $[Y; S_{1} = I_{x} \times T_{x}]$ is a left C-wrpp semigroup and $\Lambda = [Y; \Lambda_{y}; \varphi_{x, y}]$ is a right normal band, then $(i, a)^{+} = (i, 1_{a})$ for $(i, a) \in S_{a}$, where $1_{x}$ is the identity in $T_{x}$. We easily verify that $S$ is a strongly wrpp semigroup with $((i, a), \lambda) = ((i, 1_{a}), \lambda)$, and whence we can also check that $S$ is a weakly left C-wrpp semigroup.

Weakly left C-semigroups were first investigated by Guo [6] in 1996, and weakly left C-rpp semigroups were investigated by Cao [9] in 2000. It is clear that weakly left C-semigroups and weakly left C-rpp semigroups are special weakly left C-wrpp semigroups. As applications of Theorem 3.4, we have the following corollaries:

Corollary 3.5 A semigroup $S$ is a weakly left C-wrpp semigroup if and only if $S$ is isomorphic to a spined product $S_{1} \times \Lambda$ of a left C-wrpp semigroup $S_{1} = [Y; S_{1} = I_{x} \times T_{x}]$ and a right normal band $\Lambda = [Y; \Lambda_{y}; \varphi_{x, y}]$ with respect to semilattice $Y$.

Corollary 3.6 A semigroup $S$ is a weakly left C-semigroup if and only if $S$ is isomorphic to a spined product $S_{1} \times \Lambda$ of a left C-semigroup $S_{1} = [Y; S_{1} = I_{x} \times T_{x}]$ and a right normal band $\Lambda = [Y; \Lambda_{y}; \varphi_{x, y}]$ with respect to semilattice $Y$.

REFERENCES