**Abstract**—The purpose of the present paper is to study the concept of fuzzy bi-ideals in ternary semirings. We give some characterizations of fuzzy bi-ideals. Characterizations of regular ternary semirings are provided.

**Keywords**—Fuzzy ternary subsemiring, fuzzy quasi-ideal, fuzzy bi-ideal, regular ternary semiring

## I. INTRODUCTION

**Ternary** semirings are one of the generalized structures of semirings. The notion of ternary algebraic system was introduced by Lehmer [8]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. Dutta and Kar [1] introduced the notion of ternary semiring which is a generalization of the ternary ring introduced by Lister [9]. Good and Hughes [3] introduced the notion of bi-ideal and Steinfeld [11], [12] introduced the notion of quasi-ideal. In 2005, Kar [5] studied quasi-ideals and bi-ideals of ternary semirings. Ternary semiring arises naturally, for instance, the ring of integers \( Z \) is a ternary semiring. The subset \( Z^+ \) of all positive integers of \( Z \) forms an additive semigroup and which is closed under the ring product. Now, if we consider the subset \( Z^- \) of all negative integers of \( Z \), then we see that \( Z^- \) is closed under the binary ring product; however, \( Z^- \) is not closed under the binary ring product, i.e., \( Z^- \) forms a ternary semiring. Thus, we see that in the ring of integers \( Z \), \( Z^+ \) forms a semiring whereas \( Z^- \) forms a ternary semiring. More generally; in an ordered ring, we can see that its positive cone forms a semiring whereas its negative cone forms a ternary semiring. Thus a ternary semiring may be considered as a counterpart of semiring in an ordered ring.

The theory of fuzzy sets was first inspired by Zadeh [14]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics. Rosenfeld [13] introduced fuzzy sets in the realm of group theory. Fuzzy ideals in rings were introduced by Liu [10] and it has been studied by several authors. Jun [4] and Kim and Park [7] have also studied fuzzy ideals in semirings. In 2007, [6] we have introduced the notions of fuzzy ideals and fuzzy quasi-ideals in ternary semirings. Our main purpose in this paper is to introduce the notions of fuzzy bi-ideal in ternary semirings and study regular ternary semiring in terms of these two subsystems of fuzzy subsemirings. We give some characterizations of fuzzy bi-ideals.

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## II. PRELIMINARIES

In this section, we review some definitions and some results which will be used in later sections.

**Definition 2.1.** A set \( R \) together with associative binary operations called addition and multiplication (denoted by + and \cdot , respectively) will be called a semiring provided:

(i) Addition is a commutative operation.
(ii) there exists \( 0 \in R \) such that \( a + 0 = a \) and \( a \cdot 0 = 0 \) for each \( a \in R \).
(iii) multiplication distributes over addition both from the left and the right. i.e., \( a(b + c) = ab + ac \) and \( (a + b)c = ac + bc \).

**Definition 2.2.** A nonempty set \( S \) together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if \( (S, +) \) is an additive commutative semigroup satisfying the following conditions:

(i) \( (abc)de = a(bcd)e = ab(cde) \)
(ii) \( (a + b)cd = acd + bcd \)
(iii) \( a(b + c)d = abd + acd \)
(iv) \( ab(c + d) = abc + abd \), for all \( a, b, c, d, e \in S \).

**Definition 2.3.** (i) Let \( S \) be a ternary semiring. An additive subsemigroup \( T \) of \( S \) is called a ternary semiring if \( t_1t_2t_3 \in T \), for all \( t_1, t_2, t_3 \in T \).
(ii) Let \( S \) be a ternary semiring. If there exists an element \( 0 \in S \) such that \( 0+a = a \) and \( 0ab = a0b = 0b = 0a = 0 \) for all \( a, b \in S \), then “0” is called the zero element or simply the zero of the ternary semiring \( S \). In this case we say that \( S \) is a ternary semiring with zero.
(iii) Let \( A, B, C \) be three subsets of ternary semiring \( S \). Then by \( ABC \), we mean the set of all finite sums of the form \( \sum a_ib_jc_k \) with \( a_i \in A, b_j \in B, c_k \in C \).
(iv) An additive subsemigroup \( I \) of \( S \) is called a left (resp., right, and lateral) ideal of \( S \) if \( s_1s_2i \) (resp. \( is_1s_2, s_1is_2 \)) \( \subseteq I \), for all \( s_1, s_2 \in S \) and \( i \in I \). If \( I \) is both left and right ideal of \( S \), then \( I \) is called a two-sided ideal of \( S \). If \( I \) is a left, a right and a lateral ideal of \( S \), then \( I \) is called an ideal of \( S \). An ideal of \( I \) of \( S \) is called a proper ideal if \( I \neq S \).

**Definition 2.4.** (i) An additive subsemigroup \( (Q, +) \) of a ternary semiring \( S \) is called a quasi-ideal of \( S \) if \( QSS \cap (SQS + SSQ) \subseteq Q \).
(ii) An additive subsemigroup \( (Q, +) \) of a ternary semiring \( S \) is called a bi-ideal of \( S \) if \( QSQSQ \subseteq Q \).

Now, we review the concept of fuzzy sets [10], [13], [14]). Let \( X \) be a non-empty set. A map \( \mu : X \rightarrow [0, 1] \) is called a fuzzy set in \( X \), and the complement of a fuzzy set \( \mu \) in \( X \),
denoted by \( \mathfrak{F} \), is the fuzzy set in \( X \) given by \( \mathfrak{F}(x) = 1 - \mu(x) \) for all \( x \in X \).

Let \( X \) and \( Y \) be two non-empty sets and \( f : X \to Y \) a function, and let \( \mu \) and \( \nu \) be any fuzzy sets in \( X \) and \( Y \) respectively. The image of \( \mu \) under \( f \), denoted by \( f(\mu) \), is a fuzzy set in \( Y \) defined by:

\[
f(\mu)(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\
0 & \text{otherwise,}
\end{cases}
\]

for each \( y \in Y \). The preimage of \( \nu \) under \( f \), denoted by \( f^{-1}(\nu) \), is a fuzzy set in \( X \) defined by \( (f^{-1}(\nu))(x) = \nu(f(x)) \) for each \( x \in X \).

**Definition 2.5.** A fuzzy ideal of a semiring \( R \) is a function \( A : R \to [0, 1] \) satisfying the following conditions:

(i) \( A \) is a fuzzy subsemigroup of \((R,+)\); i.e., \( A(x - y) \geq \min\{A(x), A(y)\} \),

(ii) \( A(xy) \geq \max\{A(x), A(y)\} \), for all \( x, y \in R \).

**Definition 2.6.** Let \( A \) and \( B \) be any two subsets of \( S \). Then \( A \cap B, A \cup B, A+B \) and \( A \circ B \) are fuzzy subsets of \( S \) defined by:

\[
(A \cap B) = \min\{A(x), B(x)\},
\]
\[
(A \cup B) = \max\{A(x), B(x)\},
\]
\[
(A + B)(x) = \begin{cases} 
\sup\{\min\{A(y), A(z)\} \} & \text{if } x = y + z, \\
0 & \text{otherwise.}
\end{cases}
\]
\[
(A \circ B)(x) = \begin{cases} 
\sup\{\min\{A(y), A(z)\} \} & \text{if } x = yz, \\
0 & \text{otherwise.}
\end{cases}
\]

For any \( x \in S \) and \( t \in (0, 1] \), define a fuzzy point \( x_t \) as:

\[
x_t(y) = \begin{cases} 
t & \text{if } y = x, \\
0 & \text{if } y \neq x.
\end{cases}
\]

If \( x_t \) is a fuzzy point and \( A \) is any fuzzy subset of \( S \) and \( x_t \leq A \), then we write \( x_t \in A \). Note that \( x_t \in A \) if and only if \( x_t \in A \) where \( A_t \) is a level subset of \( A \). If \( x_t \) and \( y_t \) are fuzzy points, then \( x_t + y_t = (xy)_{\min\{r,s\}} \).

**Definition 2.7.** [6] A fuzzy subset \( A \) of a fuzzy subsemigroup of \( S \) is called a fuzzy ternary subsemiring of \( S \) if:

(i) \( A(x - y) \geq \min\{A(x), A(y)\} \), for all \( x, y \in S \)

(ii) \( A(-x) = A(x) \)

(iii) \( A(xy) \geq \min\{A(x), A(y)\} \), for all \( x, y, z \in S \).

**Definition 2.8** [6] A fuzzy subsemigroup \( A \) of a ternary semiring \( S \) called a fuzzy ideal of \( S \) if \( A : S \to [0, 1] \) satisfying the following conditions:

(i) \( A(x - y) \geq \min\{A(x), A(y)\} \), for all \( x, y \in S \)

(ii) \( A(xy) \geq \max\{A(x), A(y)\} \)

(iii) \( A(xy) \geq A(x) \)

(iv) \( A(xy) \geq A(y) \), for all \( x, y, z \in S \)

A fuzzy subset \( A \) with conditions (i) and (ii) is called a fuzzy left ideal of \( S \). If \( A \) satisfies (i) and (iii), then it is called a fuzzy right ideal of \( S \). Also if \( A \) satisfies (i) and (iv), then it is called a fuzzy lateral ideal of \( S \). A fuzzy ideal is a ternary semiring of \( S \), if \( A \) is a fuzzy left, a fuzzy right and a fuzzy lateral ideal of \( S \).

**Example 2.9** [6]. Let \( Z \) be a ring of integers and \( S = \mathbb{Z}_0 \subseteq \mathbb{Z} \) be the set of all negative integers with zero. Then with the binary addition and ternary multiplication, \((\mathbb{Z}_0, +, \cdot)\) forms a ternary semiring \( S \) with zero. Define a fuzzy subset \( A : Z \to [0, 1] \), we have:

\[
A(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Z}_0, \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( A \) is a fuzzy ternary subsemiring of \( S \).

**Example 2.10** [6]. Consider the set integer module 5, non-positive integer \( \mathbb{Z}_5 = \{0, -1, -2, -3, -4\} \) with the usual addition and ternary multiplication, we have:

\[
\begin{array}{cccccccc}
+ & 0 & -1 & -2 & -3 & -4 & 0 & 0 \\
0 & 0 & -1 & -2 & -3 & -4 & 0 & 0 \\
-1 & -1 & 0 & -1 & -2 & -3 & -4 & 0 \\
-2 & -2 & -1 & 0 & -1 & -2 & -3 & -4 \\
-3 & -3 & -2 & -1 & 0 & -1 & -2 & -3 \\
-4 & -4 & -3 & -2 & -1 & 0 & -1 & -2 \\
\end{array}
\]

Clearly \((\mathbb{Z}_5, +, \cdot)\) is a ternary semiring. Let a fuzzy subset \( A : \mathbb{Z}_5 \to [0, 1] \) be defined by \( A(0) = t_0 \) and \( A(-1) = A(-2) = A(-3) = A(-4) = t_1 \), where \( t_0 \leq t_1 \) and \( t_0, t_1 \in (0, 1] \). Routine calculations show that \( A \) is a fuzzy ideal of \( \mathbb{Z}_5 \).

**Definition 2.11** [6] Let \( A \) be a fuzzy subset of ternary semiring \( S \). We define:

\[
SAS + SSASS(z) = \begin{cases} 
\sup\{\min\{A(a), A(b)\} \} & \text{if } z = x(a + xy)g, \\
0 & \text{otherwise.}
\end{cases}
\]

for all \( x, y, a, b \in S \).

## III. Fuzzy Bi-Ideals of Ternary Semiring

**Definition 3.1.** A fuzzy subsemigroup \( \mu \) of a ternary semiring \( S \) is called a fuzzy quasi-ideal of \( S \) if:

\[
(FQI1) \mu S \cap S \mu \subseteq \mu, \quad (FQI2) \mu S \cap SS\mu \subseteq \mu,
\]

i.e., \( \mu(x) \geq \min\{\mu(S), \mu(S)S, SS\mu\} \).
To strengthen the above definition, we present the following example.

**Example 3.2.** Consider the ternary semiring \((\mathbb{Z}_5, +, \cdot, \cdot)\) as defined in Example 2.10 in this paper. Let \(A = \{0, -2, -3\} \). Then \(SSA = \{-2, -3, 4\}, (SS + SSASS) = \{0, 1, -2, -3\} \) and \(ASS = \{-1, -2, -3\} \). Therefore \(ASS \cap (SS + SSASS) \cap SSA = \{-2, -3\} \subseteq A \). Hence \(A\) is a quasi-ideal of \(\mathbb{Z}_5\). Define a fuzzy subset \(A : \mathbb{Z}_5 \to [0, 1] \) by \(A(0) = A(-2) = A(-3) = 1\) and \(A(-1) = A(-4) = 0\). Clearly \(A\) is a fuzzy quasi-ideal of \(\mathbb{Z}_5\).

**Definition 3.3.** A fuzzy ternary subsemiring \(\mu\) of \(S\) is called a fuzzy bi-ideal of \(S\) if

\[
\mu(S)\mu(S) \subseteq \mu(x, y, z, w, v) \quad \forall \quad x, y, z, w, v \in S
\]

i.e., \(\mu(x,y,z) \geq \min\{\mu(x), \mu(y), \mu(z)\} \quad \forall \quad x, y, z \in S\). Then \(\mathbb{Z} = S\) is a ternary semiring under usual addition and ternary multiplication. Let \(B = 2S\) Thus \(BBSS = 2SS2SS2S = 6(\text{SSS})SS = 6(\text{SSS}) = 6S \subseteq 2S = B\). Hence \(B\) is a bi-ideal of \(\mathbb{Z}^+\).

**Definition 3.7.** Let \(\mathbb{Z} = S\) be the set of all negative integers. Then \(\mathbb{Z} = S\) is a ternary semiring under usual addition and ternary multiplication. Then \(B = 2S\) Thus \(BBSS = 2SS2SS2S = 6(\text{SSS})SS = 6(\text{SSS}) = 6S \subseteq 2S = B\). Hence \(B\) is a bi-ideal of \(\mathbb{Z}^+\).

Denote \(\mu : S \to [0, 1]\) by

\[
\mu(x) = \begin{cases} 
1, & \text{if } x \in 2S \\
0, & \text{otherwise}
\end{cases}
\]

For any \(t \in [0, 1]\), \(\mu_t = \{2S\}\), since \(2S\) is a bi-ideal in \(\mathbb{Z}^+\), \(\mu_t\) is the bi-ideal in \(\mathbb{Z}^+\) for all \(t\). Hence \(\mu\) is a fuzzy bi-ideal of \(\mathbb{Z}^+\).

**Lemma 3.5.** Let \(\mu\) be a fuzzy subset of \(S\). If \(\mu\) is a fuzzy left ideal, fuzzy right ideal and left ideal of ternary semiring of \(S\), then \(\mu\) is a fuzzy quasi-ideal of \(S\).

**Proof:** Let \(\mu\) be a fuzzy left ideal, fuzzy right ideal and fuzzy left ideal of \(S\). Let \(x = a_1s_2 = a_1(1 + c_1c_2)s_2 = s_1s_2d\) where \(a, b, c, d, s_1, s_2 \in S\).

Consider \((\mu S) \cap (S\mu S + SS\mu S) \cap SS\mu(x)\)

\[
= \min\{\mu(x), \mu_S(x)S + SS\mu(x)\}
\]

\[
= \min\{\mu_S(x), \mu_S(x)S + SS\mu(x)\}
\]

\[
= \min\{\mu_S(x), SS\mu(x)\}
\]

\[
= \mu_S(x)S + SS\mu(x)\subseteq S \subseteq \mu_S(x)S + SS\mu(x)
\]

Thus, \((\mu S) \cap (S\mu S + SS\mu S) \cap SS\mu(x) \subseteq \mu(x)\).

Hence \(\mu\) is a fuzzy quasi-ideal of \(S\).

**Lemma 3.6.** For any non-empty subsets \(A, B\) and \(C\) of \(S\),

\[
1. f_{A\cap B}fc = f_{A\cup B\cap C}
\]

\[
2. f_A \cap f_B \cap f_C = f_{A\cup B\cup C}
\]

\[
3. f_A + f_B = f_{A\cup B}
\]

**Proof:** Proof is straight forward.

**Lemma 3.7.** Let \(Q\) be an additive subsemigroup of \(S\).

\[
1. Q\) is a quasi-ideal of \(S\) if and only if \(f_Q\) is a fuzzy quasi-ideal of \(S\).
\]

\[
2. Q\) is a bi-ideal of \(S\) if and only if \(f_Q\) is a fuzzy bi-ideal of \(S\).
\]

**Proof:** Proof of (1) can be seen in [8].

**Lemma 3.8.** Any fuzzy quasi-ideal of \(S\) is a fuzzy bi-ideal of \(S\).

**Proof:** Let \(\mu\) be any fuzzy quasi-ideal of \(S\). Then we have

\[
\mu S\mu \subseteq \mu S\mu \subseteq \mu S\mu S \subseteq \mu S\mu \subseteq \mu S\mu S
\]

so, \(\mu S\mu \subseteq \mu S\mu S\mu\) and taking \(\{0\} \subseteq S \subseteq \mu S\mu\) we have, \(\mu S\mu \subseteq \mu S\mu \cap (S\mu S + SS\mu S) \cap SS\mu \subseteq S\mu\)

Hence, \(\mu\) is a fuzzy bi-ideal of \(S\).

**Remark 3.9.** The converse of Lemma 3.8 does not hold, in general, that is, a fuzzy bi-ideal of a ternary semiring \(S\) may not be a fuzzy quasi-ideal of \(S\).

**Theorem 3.10.** Let \(\mu\) be a fuzzy subset of \(S\). If \(\mu\) is a fuzzy left, fuzzy right and left ideal of ternary semiring of \(S\), then \(\mu\) is a fuzzy bi-ideal of \(S\).

**Proof:** As \(\mu\) is fuzzy left, right and left ideal of \(S\) and Lemma 3.5, \(\mu\) is a fuzzy quasi-ideal of \(S\). Hence by Lemma 3.8, \(\mu\) is a fuzzy bi-ideal of \(S\).

**Theorem 3.11.**[6] Let \(\mu\) be a fuzzy subset of \(S\). Then \(\mu\) is a fuzzy quasi-ideal of \(S\), if and only if \(\mu_t\) is a quasi-ideal of \(S\), for all \(t \in 1m(\mu)\).

**Proof:** Let \(\mu\) be a fuzzy bi-ideal of \(S\). Let \(t \in 1m(\mu)\). Suppose \(x, y, z \in S\) such that \(x, y, z \in \mu_t\). Then \(\mu(x) \geq t, \mu(y) \geq t, \mu(z) \geq t\)
ideal of ternary semiring

Then

ternary subsemiring of

and

Theorem 3.14.

Let

as a fuzzy bi-ideal of

as a fuzzy bi-ideal of a ternary semiring

Consider

Since any fuzzy quasi-ideal of

(2)

First assume that (1) holds. Let

For a ternary semiring

Lemma 3.7 (1).

Theorem 3.14. If

is a fuzzy bi-ideal of a ternary semiring

and

is a fuzzy ternary subsemiring of

then

is a fuzzy bi-ideal of

Proof: Let

be a fuzzy bi-ideal and

be a fuzzy ternary subsemiring of

Clearly

is a fuzzy ternary subsemiring of

Next we prove that

is a fuzzy bi-ideal of ternary semiring

Let

such that

Theorem 4.1.

A ternary semiring

is called regular if for every

there exists an

such that

Lemma 4.1. A ternary semiring

is regular if and only if

for every fuzzy right ideal

fuzzy left ideal

and

lateral ideal

of

Proof: Straight forward from Theorem 5.1 in [5].

IV. REGULAR TERNARY SEMIRING

A ternary semiring

is called regular if for every

there exists an

such that

Then since

is regular, there exists an element

in

such that

Then we have

Theorem 4.2. For a ternary semiring

the following conditions are equivalent:

(1) 

is regular

(2) 

for every fuzzy bi-ideal

of

(3) 

for every fuzzy quasi-ideal

of

Proof: (1)⇒(2) First assume that (1) holds. Let

be any fuzzy bi-ideal of

and

any element of

Then since

is regular, there exists an element

in

such that

Then we have

(2)⇒(3) Since any fuzzy quasi-ideal of

is a fuzzy bi-ideal of

by Lemma 3.8.

(3)⇒(1) Assume (3) holds. Let

be any quasi-ideal of

and

any element of

Then it follows from Lemma 3.7 (1)
that the characteristic function $f_Q$ is a quasi-ideal of $S$. Then we have

$$f_Q(a) = (f_Q + f_S + f_Q + f_S + f_Q)(a) = f_Q(a) = 1$$

and so, $a \in QS$. Thus $Q \subseteq QS$. On the other hand, $Q$ is a quasi-ideal of $S$.

$$QS \subseteq (QS \cap QS \cap SS)$$

then,

$$QS \subseteq (QS \cap (SQ + SS) \cap SS) \cap SS \subseteq Q$$

and so we have $QS = Q$ and hence, by [5, Theorem 3.4], $S$ is a regular ternary semiring.

References


