A new time discontinuous expanded mixed element method for convection-dominated diffusion equation

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Abstract—In this paper, a new time discontinuous expanded mixed finite element method is proposed and analyzed for two-order convection-dominated diffusion problem. The proofs of the stability of the proposed scheme and the uniqueness of the discrete solution are given. Moreover, the error estimates of the scalar unknown, its gradient and its flux in the \(L^\infty(J, L^2(\Omega))\)-norm are obtained.

Keywords—Convection-dominated diffusion equation; Expanded mixed method; Time discontinuous scheme; Stability; Error estimates.

I. INTRODUCTION

In this paper, we consider the following convection-dominated diffusion equation

\[
\begin{align*}
\frac{du}{dt} - \nabla \cdot (a(x,t)\nabla u) + b \cdot \nabla u(x,t) &+ cu(x,t) = f(x,t), \Omega \times J \\
u(x,t) &= 0, \partial \Omega \times J, \\
u(x,0) &= u_0(x), \Omega,
\end{align*}
\]

where \(\Omega\) is a bounded convex polygonal domain in \(\mathbb{R}^d (d = 1, 2, 3)\) with Lipschitz continuous boundary \(\partial \Omega\), \(J = (0,T)\) is the time interval with \(0 < T < \infty\). \(u_0(x)\) and \(f(x,t)\) are given functions, coefficients \(a = a(x,t)\) and \(c = c(x,t)\) are two smooth and bounded functions, coefficient \(b(x) = (\hat{b}_1(x), \ldots , \hat{b}_d(x))\) is a bounded vector, and \(|b| = (\sum_{i=1}^{d} \hat{b}_i^2)^{\frac{1}{2}} \leq 1\).


In 1997, a expanded mixed finite element method was proposed and analysed by Arbogast et al. [11]. And some mathematical theories were given and proved by Chen [12] for second-order linear elliptic equation, [13] for second-order quasilinear elliptic equation and [14] for fourth-order elliptic problems. With the development of the expanded mixed finite element method, the method were applied to many evolution equations. In [15], some error estimates of the expanded mixed element for a kind of parabolic equation were given. Woodward and Dawson [16] studied the expanded mixed finite element method for nonlinear parabolic equation. In [17], a posteriori error estimator for expanded mixed hybrid methods was proposed. Chen et al. [18] studied a two-grid method for expanded mixed finite-element solution of semilinear reaction-diffusion equations. In [19], a two-grid method with expanded mixed method was studied for nonlinear reaction-diffusion equations. Song and Yuan [20] proposed the expanded upwind-mixed multi-step method for the miscible displacement problem in three dimensions. Guo and Chen [21] developed and analysed an expanded characteristic-mixed finite element method for a convection-dominated transport problem. In 2010, Chen and Wang [22] proposed an \(H^1\)-Galerkin expanded mixed method for a nonlinear parabolic equation in porous medium flow and Liu and Li [23] studied the \(H^1\)-Galerkin expanded mixed method for pseudo-hyperbolic equation. Liu [24], studied the \(H^1\)-Galerkin expanded mixed method for nonlinear viscoelasticity-type equation. In [25] and [26], the expanded mixed covolume method was studied for the linear integro-differential equation of parabolic type and elliptic problems, respectively. Jiang and Li [27] studied an expanded mixed semidiscrete scheme for the problem of purely longitudinal motion of a homogeneous bar.

In this article, we will develop a new expanded mixed finite element method based time discontinuous finite element method [6], [7], [8], [9], [10], prove the stability and uniqueness for discrete scheme, and obtain the error estimates. In the near future, we will study the space-time discontinuous expanded mixed finite element method for some evolution equations.

II. NOTATIONS AND DEFINITIONS

In order to introduce the mixed time discontinuous space-time finite element method for equation (1), we discretize the
time interval $[0, T]$ by $0 = t^0 < t^1 < \cdots < t^N = T$ firstly. Let $I_n = (t^n, t^{n+1})$, time step $\kappa_n = t^{n+1} - t^n$, $n = 0, 1, 2, \cdots, N - 1$. $\mathcal{T}_n$ is the regular partition of $\Omega$ and the partition unit is $\tau$. Define the space-time domain $Q := \Omega \times J$, the space-time slab $S^n := \Omega \times I_n$. Suppose $\mathcal{T}_n$ is the regular partition of $S^n$ and the partition unit is $K = \tau \times \mathcal{T}_n$. Let $h_n = \max_{K \in \mathcal{T}_n} (h_K)$, $n = 0, 1, 2, \cdots, N - 1$, $h = \max_nh_n$. Define discrete approximate spaces 

\[ W_{h,n} = \{ v : v|_K \in P_k(\tau) \times P_k(I_n), \forall K \in \mathcal{T}_n \}, \]

\[ Q_{h,n} = \{ v : v|_K \in P_0(\tau) \times P_0(I_n), \forall K \in \mathcal{T}_n \}, \]

\[ V_{h,n} = \{ \varphi : \varphi|_K \in (Q_{h,n})^d, \nabla \cdot \varphi|_K \in Q_{h,n}, \forall K \in \mathcal{T}_n \}, \]

\[ A_{h,n} = \{ \varphi : \varphi|_K \in (Q_{h,n})^d, \varphi|_K \text{ cut off at } K \}, \]

where $P_k$ denotes the polynomial space of degree at most $k$. We introduce definitions and lemmas which will be used in this paper.

**Definition 1:** Define the inner product of space-time slab $S^n$ by $(\omega, v)_n = (\omega, v)_\Omega \int_{I_n} (\omega, v)ds$, where the inner product in $\Omega$ and the corresponding norm is $||v|| = (v, v)_\Omega^{1/2}$.

**Definition 2:** At the time level $t = t^n$ $(n = 0, 1, \cdots, N - 1)$, define the inner product of $L_2$ by $(\omega, v)_n = (\omega(t^n), v(t^n))_\Omega$ and the corresponding norm is $||v|| = (v, v)_\Omega^{1/2}$.

**Definition 3:** Define the left and right limits by $v_\pm(x, t) = \lim_{\epsilon \to 0} v(x, t + \epsilon)$ and the jump term at discontinuous nodes $s \in [0, T)$ in time by $[v] = v_+ - v_-$. Define norm $||v||_2 = \frac{1}{2} \sum_{n=1}^{N-1} [|v|_n^2] + |v|_0^2$.

**Definition 4:** Define the norm of space $L_2(J, L_2(\Omega))$ by $||v||_Q^2 = \int_0^T ||v||_\Omega^2 dt$.

**Definition 5:** Define the norm of space $L_\omega(\bar{J}, L_2(\Omega))$ by $\max_{t \in [0, T]} \int_\Omega |v|^2 dx$, where $\| \cdot \|$ denotes the corresponding norm of Sobolev space $L_2(\Omega)$.

**Definition 6:** $V = H(div, \Omega) = \{ v \in (L_2(\Omega))^d \mid \nabla \cdot v \in L_2(\Omega) \}$, with norm $\| v \|^2 = \| v \|^2 + \| \nabla \cdot v \|^2$, $W = L_2(\Omega)$ or $W = \{ w \in L_2(\Omega) \mid \partial \Omega = 0 \}$, $\Lambda = (L_2(\Omega))^d$.

**III. THE EXISTENCE, UNIQUENESS AND STABILITY FOR SEMI-DISCRETE SCHEME**

Introducing the two auxiliary variables $\lambda = -\nabla u$ and $\sigma = -a(x, t)\nabla u = a\lambda$, we obtain the following first-order system for $(1)$

\[
\begin{align*}
(a) \quad & u_t + \nabla \cdot b + \lambda + cu = f, \quad (x, t) \in \Omega \times J, \\
(b) \quad & \lambda + \nabla u = 0, \quad (x, t) \in \Omega \times J, \\
(c) \quad & \sigma \cdot a\lambda = 0, \quad (x, t) \in \Omega \times J, \\
(d) \quad & u(x, t) = 0, \quad (x, t) \in \partial \Omega \times J, \\
(e) \quad & u(x, 0) = u_0(x), \quad x \in \Omega.
\end{align*}
\]

The new time discontinuous expanded mixed weak formulation of (2) is as follows

\[
\begin{align*}
(a) \quad & \int_0^T (\omega, u)_0 + \int_0^T (\nabla \cdot \sigma, w)_0 dt = \int_0^T (b \cdot \lambda, w)_0 dt + \sum_{n=1}^{N-1} \int_0^{t^n} (|u|, w_n)_0 + \langle u_+, w_n \rangle_0 + \int_0^{t^n} (c u, w)_0 dt, \\
(b) \quad & \int_0^T (\lambda, v)_0 + \int_0^T (u, \nabla \cdot v)_0 dt = 0, \forall v \in V, t < J, \\
(c) \quad & \int_0^T (a\lambda, \mu)_0 - \int_0^T (\sigma, \mu)_0 = 0, \forall \mu \in \Lambda, t < J.
\end{align*}
\]

Then, the semi-discrete mixed finite element scheme for (4) is to determine $(\sigma^h, \lambda^h, u^h) \in V_{h,n} \times \Lambda_{h,n} \times W_{h,n}$ such that

\[
\begin{align*}
(a) \quad & (u^h, w_n) + (\nabla \cdot \sigma^h, w_n) - (b \cdot \lambda^h, w_n) \\
& + \langle |u|, w_n \rangle + \langle cu, w_n \rangle = (f, w)_n, \\
(b) \quad & (\lambda^h, v)_n - (u^h, \nabla \cdot v)_n = 0, \\
(c) \quad & (a\lambda^h, \mu)_n - (\sigma^h, \mu)_n = 0.
\end{align*}
\]

We will prove the stability for semi-discrete scheme (5).

**Theorem 3.1:** The semi-discrete scheme (5) is stable and holds the following inequality

\[
\max_{0 \leq t < T} \| u^h \|_\Omega + || \sigma^h ||_\Omega + || \lambda^h ||_\Omega \leq M(\| u_0 \|_\Omega + || f ||_Q),
\]

where $M$ is a constant independent of $h_n$ and $\kappa_n$.

**Proof:** Choosing $w^h = u^h, \sigma^h = \sigma^h, \mu^h = \lambda^h$ in (5) and summing from $n = 1$ to $N$, we have

\[
\begin{align*}
\int_0^T (u^h, u^h)_0 + \int_0^T (u^h, \nabla \cdot \sigma^h)_0 dt & - \int_0^T (b \cdot \lambda^h, u^h)_0 dt + \sum_{n=1}^{N-1} \int_0^{t^n} (|u|, u^h)_0 + \langle u^h, u^h \rangle_0 + \int_0^{t^n} (c u^h, u^h)_0 dt \\
& = (u^h, u^h)_0 + \int_0^T (f, u^h)_0 dt,
\end{align*}
\]

\[
\begin{align*}
& \int_0^T (\lambda^h, u^h)_0 - \int_0^T (u^h, \nabla \cdot \sigma^h)_0 dt = 0, \\
& \int_0^T (a\lambda^h, \lambda^h)_0 - \int_0^T (\sigma^h, \lambda^h)_0 = 0.
\end{align*}
\]
Adding the three equations, we obtain
\[
\int_0^T (u_h^t, u_h^b) dt + \int_0^T (a \lambda_h^t, \lambda_h^b) dt - \int_0^T (b \cdot \lambda_h^b, u_h^b) dt \\
+ \sum_{n=1}^{N-1} (\langle u_h^n, u_h^n \rangle_n + (u_h^n, u_h^n)_n) + \int_0^T (c u_h^b, u_h^b) dt \\
= (u_h^b, u_h^b)_0 + \int_0^T (f, u_h^b) dt.
\]
(7)

Integration by parts for the first term in (7), we have
\[
\int_0^T (u_h^b)^2 dt + \int_0^T (a \lambda_h^t, \lambda_h^b) dt + \int_0^T (c u_h^b, u_h^b) dt \\
= (u_h^b, u_h^b)_0 + \int_0^T (f, u_h^b) dt + \int_0^T (b \cdot \lambda_h^b, u_h^b) dt.
\]
(8)

Using Cauchy-Schwarz inequality and Young inequality, we obtain
\[
||u_h^b||^2 + \int_0^T ||\lambda_h^b||^2 dt + \int_0^T ||u_h^b||^2 dt \\
\leq (u_h^b, u_h^b)_0 + \int_0^T ||f||^2 dt.
\]
(9)

Take \( \mu_h^b = \sigma_h^b \) in (5c) and use Young inequality to get
\[
\int_0^T ||\sigma_h^b||^2 dt \leq C \int_0^T ||\lambda_h^b||^2 dt.
\]
(10)

Combing (9) and (10) and taking the max norm with respect to time \( t \), we get the conclusion for theorem 3.1.

**Theorem 3.2:** There exists a unique discrete solution to the semi-discrete scheme (5).

**Proof:** In fact, since (5) is linear, it suffices to show that the associated homogeneous system
\[
\begin{align*}
(a) \quad & (u_h^t, u_h^b)_n + (\nabla \cdot \sigma_h^b, w_h^b)_n - (b \cdot \lambda_h^b, w_h^b)_n, \\
(b) \quad & (\lambda_h^b, \lambda_h^b)_n = 0, \\
(c) \quad & (\sigma_h^b, \mu_h^b)_n = 0, \\
(d) \quad & (u_h^b(0), w_h^b) = 0,
\end{align*}
\]
(11)

has only the zero trivial solution.

Taking \( w_h = u_h, v_h = \sigma_h, \mu_h = \lambda_h \) in (11) and adding the three equations, we get
\[
|u_h^b|_n^{l+1} - |u_h^b|_n^{l} + 2|\nabla \lambda_h^b|_n^{l} \\
+ 2|u_h^b|_n^{l} + 2|\nabla u_h^b|_n^{l} = (b \cdot \lambda_h^b, u_h^b)_n.
\]
(12)

Using Cauchy-Schwarz inequality, we have
\[
|u_h^b|_n^{l+1} + |u_h^b|_n^{l} + 2|\nabla u_h^b|_n^{l} = (b \cdot \lambda_h^b, u_h^b)_n. \\
+ 2(c_0 - 2\xi_0)|u_h^b|_n^{l} \leq 0
\]
(13)
So, we have \( \lambda_h^b = 0 \) and \( u_h^b = 0 \).

Choosing \( \mu_h^b = \sigma_h^b \), in (11c), using Cauchy-Schwarz inequality and Young inequality, we have
\[
||\sigma_h^b||_n \leq C||\lambda_h^b||_n.
\]
(14)

so, we have \( \sigma_h^b = 0 \).

**IV. ERROR ESTIMATES FOR** \( L^\infty(J, L^2(\Omega)) \) **NORM**

In order to analyze the error estimates of the method, we first introduce the expanded mixed elliptic projection associated with our equations.

**Lemma 4.1:** Let \( (\tilde{u}_h, \tilde{\lambda}_h, \tilde{\sigma}_h) \in W_{h,n} \times V_{h,n} \times A_{h,n} \) be given by the following mixed relations(see Refs [12], [13], [15])
\[
\begin{align*}
(a) \quad & (\nabla \cdot (\sigma_h - \tilde{\sigma}_h), w_h)_n, \\
(b) \quad & (\lambda_h - \tilde{\lambda}_h, v_h)_n = (u_h - \tilde{u}_h, \nabla \cdot v_h)_n = 0, \\
(c) \quad & (a(\lambda_h - \tilde{\lambda}_h), \mu_h)_n - (\sigma_h - \tilde{\sigma}_h, \mu_h)_n = 0, \\
& \forall w_h \in W_{h,n}, \forall v_h \in V_{h,n}, \forall \mu_h \in A_{h,n}.
\end{align*}
\]
(15)

the corresponding approximation properties hold
\[
||\lambda - \tilde{\lambda}_h|| \leq C h^l(||\sigma||_l + ||\sigma||_l), 1 \leq l \leq k + 1,
\]
(16)

\[
||\sigma - \tilde{\sigma}_h|| \leq C h^l(||\lambda||_l + ||\lambda||_l), 1 \leq l \leq k + 1,
\]
(17)

\[
||u - \tilde{u}_h|| \leq \left\{ \begin{array}{ll}
C h^l ||u||_2, & 1 \leq l \leq k + 1, \\
C h^l ||u||_Q, & 2 \leq l \leq k + 2.
\end{array} \right.
\]
(18)

**Theorem 4.2:** Let \( (\sigma, \lambda, u) \) and \( (\sigma^h, \lambda^h, u^h) \) be the solution to (2) and (5), respectively. There exists a positive constant \( C \) independent of the spatial mesh parameters \( h_n \) and time-discretization parameter \( k_n \) such that
\[
\max_{t \in J} ||\sigma - \sigma^h||_Q + ||\lambda - \lambda^h||_Q
\leq \left\{ \begin{array}{ll}
C h^l ||u||_2, & 1 \leq l \leq k + 1, \\
C h^l ||u||_Q, & 2 \leq l \leq k + 2.
\end{array} \right.
\]
(19)

**Proof:** Let
\[
\begin{align*}
& u - u^h = u - \tilde{u}_h + \tilde{u}_h - u^h = \zeta_1 + \zeta_2, \\
& \sigma - \sigma^h = \sigma - \tilde{\sigma}_h + \tilde{\sigma}_h - \sigma^h = \eta_1 + \eta_2, \\
& \lambda - \lambda^h = \lambda - \tilde{\lambda}_h + \tilde{\lambda}_h - \lambda^h = \theta_1 + \theta_2.
\end{align*}
\]
Using (3)-(5) and Lemma, we have
\[
\begin{align*}
& \int_0^T (u_t - u_t^h, w_h^b) dt + \int_0^T (\nabla \cdot \eta_2, w_h^b) dt \\
& - \int_0^T (b \cdot \theta_2, w_h^b) dt + \sum_{n=1}^{N-1} \langle u - u_h^t, \omega_h^t \rangle_n \\
& + \langle (u + u_h^t, w_h^b)_0 + \int_0^T (c(u - u_h^t), w_h^b) dt \\
= \langle u - u_h^t, w_h^b \rangle_0, \\
& \int_0^T (\theta_2, v_h^b) dt - \int_0^T (\zeta_2, \nabla \cdot v_h^b) dt = 0, \\
& \int_0^T (a \theta_2, \mu_h^b) dt - \int_0^T (\eta_2, \mu_h^b) dt = 0.
\end{align*}
\]
Take \( u^h = \zeta_2, v^h = \eta_2, \mu^h = \theta_2 \) in (19) and add the three equations to obtain
\[
\begin{align*}
\int_0^T (u_t - u_t^h, \zeta_2) dt &+ \int_0^T (b \cdot \theta_2, \zeta_2) dt \\
&+ \sum_{n=1}^{N-1} \left( \left< u - u^h \right|, \zeta_n \right) + \left< u + u^n_h, \zeta_{n+1} \right>_0 \\
&+ \int_0^T (c(u - u^h), \zeta_2) dt + \int_0^T (a\theta_2, \theta_2) dt \\
&= \left< (u - u^h), \zeta_2 \right>_0.
\end{align*}
\]

Using the definition of \(|||\cdot|||\), the above equation can be written as
\[
\begin{align*}
|||\zeta_2|||^2 &+ \int_0^T (c\zeta_2, \zeta_2) dt + \int_0^T (c\zeta_2, \zeta_2) dt \\
&+ \int_0^T (c\zeta_2, \zeta_2) dt + \int_0^T (b \cdot \theta_2, \zeta_2) dt \\
&+ \sum_{n=1}^{N-1} \left( \left|\zeta_n \right|, \zeta_n \right) + \left|\zeta_{n+1}, \zeta_{n+1} \right>_0 \\
&+ \int_0^T (a\theta_2, \theta_2) dt = \left< (u - u^h), \zeta_2 \right>_0.
\end{align*}
\]
Integration by parts for the second term in the left hand side of (21)
\[
\begin{align*}
|||\zeta_2|||^2 &+ \frac{1}{2} \int_0^T |||\zeta_2|||^2 dt + \frac{1}{2} \int_0^T |||\theta_2|||^2 dt \\
&\leq \frac{1}{2} |||\zeta_2|||^2 + K \left( \sum_{n=1}^N |\zeta_n|^2 + ||\zeta_1||_Q^2 \right).
\end{align*}
\]
Using Cauchy-Schwarz inequality, Young inequality and the definition of \(|||\cdot|||\), we have
\[
\begin{align*}
|||\zeta_2|||^2 &+ \frac{1}{2} \int_0^T |||\zeta_2|||^2 dt + \frac{1}{2} \int_0^T |||\theta_2|||^2 dt \\
&\leq \frac{1}{2} |||\zeta_2|||^2 + ||\theta_2||_Q^2 + 2||\zeta_2||_Q^2.
\end{align*}
\]
Using the inequality (16)-(18), we obtain
\[
|||\zeta_2|||| |||\theta_2||_Q + |||\zeta_2||_Q \leq C|||u|||_I, Q.
\]
Take \( \mu^h = \eta_2 \) in (19) and use Cauchy-schwarz inequality to get
\[
|||\eta_2||_Q \leq C|||\theta_2||_Q.
\]

V. CONCLUDING REMARKS

In this article, we propose a new time discontinuous expanded mixed finite element scheme for convection-dominated diffusion equation. In the near future, the proposed method will be applied to others evolution equations such as evolution integro-differential equations.

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