Fourier spectral method for analytic continuation

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Abstract—The numerical analytic continuation of a function \( f(z) = f(x + iy) \) on a strip is discussed in this paper. The data are only given approximately on the real axis. The periodicity of given data is assumed. A truncated Fourier spectral method has been introduced to deal with the ill-posedness of the problem. The theoretical results show that the discrepancy principle can work well for this problem. Some numerical results are also given to show the efficiency of the method.

Keywords—analytic continuation, ill-posed problem, regularization method, Fourier spectral method, the discrepancy principle

I. INTRODUCTION

The problems of analytic continuation are frequently encountered in many practical applications [10], [11], [13], [14]. In general, this problem is ill-posed and several techniques have been developed for it. In this paper, we present some preliminary materials which will be used throughout the paper. Let \( \Lambda = (0, 2\pi) \) and \( \Omega = \{ z = x + iy \in \mathbb{C} | x \in \mathbb{R}, |y| \leq y_0, y_0 \) is a positive constant \}.

where

\[
\|v\|_{L^p} = \left( \int_{\Lambda} |v(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty.
\]

In particular, \( L^2(\Lambda) \) is a Hilbert space with the inner product

\[
\langle u, v \rangle_{L^2(\Lambda)} = \int_{\Lambda} u(x)\overline{v(x)} \, dx.
\]

For simplicity, denote the norm \( ||v||_{L^2} \) by \( ||v||_2 \).

The Parseval equality holds, namely

\[
||v||^2 = 2\pi \sum_{l=-\infty}^{\infty} |\hat{v}_l|^2.
\]

Now let \( N \) be any positive integer and \( V_N \) be the set of all trigonometric polynomials of degree at most \( N \), i.e.,

\[
V_N = \text{span} \{ e^{ilx} | l \leq N \}.
\]

The \( L^2 \) -orthogonal projection \( P_N : L^2(\Omega) \rightarrow V_N \) is such a mapping that for any \( v \in L^2(\Omega) \),

\[
\langle v - P_N v, \phi \rangle = 0, \quad \forall \phi \in V_N.
\]

Indeed,

\[
P_N v = \sum_{|l| \leq N} \hat{v}_l e^{ilx}.
\]

We assume that

\[
f(x + iy) \in L^2(\Lambda) \quad \text{for } |y| \leq y_0.
\]

Because the function \( f(x) \) is analytic in \( \Omega \), the following series converges in \( \Omega \):

\[
f(z) = f(x + iy) = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} D^nf(x)
\]

where \( D^n = \frac{d^n}{dz^n} \). If the data \( f(x + iy) \in L^2(\Lambda) \) for all \( y \), \( 0 \leq x \leq y_0 \) and we let

\[
f_l = \frac{1}{2\pi} \int_{\Lambda} f(x + iy) e^{-ilx} \, dx, \quad l = 0, \pm 1, \ldots
\]
then we can get

\[
\hat{f}_l^y = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} D^n f(x) e^{-ilt} dx
\]

\[
= \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} (iy)^n D^n f(x) e^{-ilt} dx
\]

\[
= \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} (iy)^n (il)^n f(x) e^{-ilt} dx
\]

\[
= \frac{1}{2\pi} \sum_{n=0}^{\infty} \left( -\frac{y^2}{n!} \right) \int_{-\infty}^{\infty} f(x) e^{-ilt} dx
\]

\[
e^{-yl} \hat{f}_l^0
\]

that is to say

\[
f(z) = f(x + iy) = \sum_{l=-\infty}^{\infty} e^{-yl} \hat{f}_l^0 e^{ilt} =: A_y f_0
\]

(12)

**Lemma 1:** Let \( f(\cdot + iy) \in L^2(\Lambda) \), \( \forall |y| \leq y_0 \), then for any \( |y| \leq y_0 \),

\[
\| f(\cdot + iy) - A_y P_N f_0 \| \leq e^{N(y-y_0)} \left( \| f(\cdot + iy_0) \| + \| f(\cdot - iy_0) \| \right).
\]

**Proof:**

\[
\| f(\cdot + iy) - A_y P_N f_0 \|^2
\]

\[
= 2\pi \left( \sum_{l>N} |e^{-yl} \hat{f}_l^y|^2 + \sum_{l<-N} |e^{-yl} \hat{f}_l^y|^2 \right)
\]

\[
\leq 2\pi e^{2N(|y| - y_0)} \left( \sum_{l>N} |e^{l(y - y_0)}| |f_l|^2 + \sum_{l<-N} |e^{l(y + y_0)}| |f_l|^2 \right)
\]

\[
\leq e^{2N(|y| - y_0)} \left( \| f(\cdot + iy_0) \| + \| f(\cdot - iy_0) \| \right).
\]

III. THE METHOD AND CONVERGENCE RESULTS

We assume the exact data \( f_0 \) and the measured data \( f^\delta \) belong to \( L^2(\Lambda) \) and satisfies

\[
\| f_0 - f^\delta \| \leq \delta,
\]

(14)

where \( \delta > 0 \) denotes the noisy level.

In addition, note that for any ill posed problem some a priori assumption on the exact solution is needed and necessary, otherwise, the regularization of the approximate solution will not be obtained or the convergence rate can be arbitrary slow[3]. In this paper, we will assume there hold the following a priori bounds

\[
\| f(\cdot + iy_0) \| \leq E,
\]

(15)

and

\[
\| f(\cdot - iy_0) \| \leq E,
\]

(16)

We want to find a function \( \phi^\delta \) such that

\[
\lim_{\delta \to 0} \| f(\cdot + iy_0) - A_y \phi^\delta \| = 0, \quad \forall |y| \leq y_0.
\]

In the following, we propose a scheme to attain the function \( \phi^\delta \) from the perturbed data \( f^\delta \). We can give the approximate function as follows:

\[
\phi^\delta_m(x) = P_m f^\delta = \sum_{|l| \leq m} \hat{f}_l^\delta e^{ilt}
\]

(17)

where \( \hat{f}_l^\delta \) are the Fourier coefficients of \( f^\delta \) and \( m = m(\delta, f_0^\delta) \) is determined by the discrepancy principle

\[
\| (I - P_m) f^\delta \| \leq \tau \delta < \| (I - P_{m-1}) f^\delta \|
\]

(18)

with \( \tau > 1 \).

In the following, we will prove a convergence estimate.

**Theorem 2:** Suppose that \( \phi^\delta_m \) is defined by (17) and (18) with \( \tau > 1 \) and the conditions (14) and (15),(16) are hold, then for any \( |y| \leq y_0 \), we have

\[
\| f(\cdot + iy) - A_y \phi^\delta_m \| \leq E \frac{|\tau + 1| \delta^{m-\delta}}{|\delta^{m-\delta} + |\tau - 1| \delta |y_0|^\delta}.
\]

(19)

**Proof:**

Suppose \( 0 < y \leq y_0 \), the proof for the case \(-y_0 \leq y < 0 \) will be analogous. From (18), we can get

\[
\| (I - P_m) f^\delta \| \leq \| (I - P_m) f^\delta - f^\delta \| + \| (I - P_m)(f - f^\delta) \|
\]

\[
\leq (\tau + 1) \delta.
\]

And the following inequality is hold by Hölder inequality

\[
\| f(\cdot + iy) - A_y P_m f \| = \sum_{|l| > m} |e^{-yl} \hat{f}_l^y|^2
\]

\[
\leq \left( \sum_{|l| > m} e^{-2y|l|} \right)^{1/2} \left( \sum_{|l| > m} |\hat{f}_l^y|^2 \right)^{1/2}
\]

\[
\leq E \frac{|\tau + 1| \delta^{m-\delta}}{|\delta^{m-\delta} + |\tau - 1| \delta |y_0|^\delta}.
\]

(20)

Hence

\[
\| f(\cdot + iy) - A_y P_m f \| \leq E \frac{|\tau + 1| \delta^{m-\delta}}{|\delta^{m-\delta} + |\tau - 1| \delta |y_0|^\delta}.
\]

(21)

On the other hand,

\[
\| P_{m-1} f - f \| \geq \| (P_{m-1} f^\delta) - (I - P_{m-1})(f - f^\delta) \|
\]

\[
\geq \| P_{m-1} f^\delta - f^\delta \| - \| (I - P_{m-1})(f - f^\delta) \|.
\]

(22)

From (18), we have

\[
\| P_{m-1} f^\delta - f^\delta \| > \tau \delta,
\]

(23)

and it is obvious that

\[
\| (I - P_{m-1})(f - f^\delta) \| \leq \delta.
\]

So we can obtain

\[
\| P_{m-1} f - f \| \geq (\tau - 1) \delta.
\]

(24)

From Lemma 1

\[
(\tau - 1) \delta \leq \| P_{m-1} f - f \| \leq 2e^{1-\delta} y_0 E.
\]

(25)

So we can obtain

\[
m \geq \frac{\ln 2E - \ln(\tau - 1) \delta}{y_0} + 1.
\]
Therefore

\[
\| f(t) - A_0 \varphi^\delta_{m} \|_{\infty} \leq E_{m}(\tau + 1) \delta^2 \delta_{m} + \| A_0 P_m f - A_0 \varphi^\delta_{m} \|_{\infty} \leq E_{m}(\tau + 1) \delta^2 \delta_{m} + \epsilon_{m} \delta.
\]

IV. NUMERICAL IMPLEMENTATION

In this section, we present numerical results of one example to check the efficiency of the method. In all the cases, the discretization knots are \( t_i = ih, i = 1, \ldots, N \), with \( N = 256 \), \( h = 1/N \) and \( \epsilon \). The perturbed discrete data are given by

\[
f_i^\delta(t_i) = f(t_i) + \epsilon_i,
\]

where \( \{\epsilon_i\}_{i=0}^{N} \) are generated by Function \( \text{randn}(N+1,1) \times \delta \) in Matlab. All examples are computed by using Matlab with parameters \( \tau = 1.01 \).

**Example 1** The function

\[ f(z) = \exp(\cos(z)), \]

is a periodic analytic function with

\[ f(z)|_{y=0} = \exp(\cos(z)) \]

The \( L^2 \)-norm relative errors and the maximum–norm relative errors are given in the tables 1, 2 to verify the theoretical results. Fig. 1 is also given to compare qualitatively the computed solutions and the exact ones. All of the results show that the new method works well.

V. CONCLUSION

In this paper the truncated Fourier spectral method is used to give a stable analytic continuation of the periodic analytic function. The theoretical and numerical results indicate that the discrepancy principle can work well if we can find a suitable approximation even for severely ill-posed problem.

### References