Optimization of Lakes Aeration Process

Mohamed Abdelwahed

Abstract—The aeration process via injectors is used to combat the lack of oxygen in lakes due to eutrophication. A 3D numerical simulation of the resulting flow using a simplified model is presented. In order to generate the best dynamic in the fluid with respect to the aeration purpose, the optimization of the injectors location is considered. We propose to adapt to this problem the topological optimization algorithm based on the studied asymptotic expansion. Finally we present some numerical results, showing the efficiency of our approach with respect to the creation of a small hole in the domain. We propose in this work a topological optimization algorithm which gives the variation of a criterion considered. We propose to adapt to this problem the topological sensitivity analysis method which gives the variation of a criterion with respect to the creation of a small hole in the domain. The main idea is to derive the topological sensitivity analysis of the physical model with respect to the insertion of an injector in the fluid flow domain. We propose in this work a topological optimization algorithm based on the studied asymptotic expansion. Finally we present some numerical results, showing the efficiency of our approach.

Keywords—Quasi Stokes equations, Numerical simulation, topological optimization, sensitivity analysis.

I. INTRODUCTION

THE mechanical aeration process in water reservoirs is one of the most used techniques to combat eutrophication. It consists on pumping a source of compressed air in the reservoir bottom via injectors in order to create a dynamic and aerate the water by bringing it in contact with the surface air. We focus in this work in the first hand to the direct problem. It concerns the numerical simulation of the resulting two phase water air-bubbles flow. Different models can be used to describe this problem [2], [4], [5], [9]. Using the fact that the fluid phase is dominant. We used a simplified model in which the water phase is characterized as the solution to a topological optimization problem. The topological sensitivity analysis is used to solve this problem [6], [7], [8], [10]. The main idea is to compute the asymptotic topological expansion with respect to the insertion of an injector.

The paper is organized as follows. The used model, its numerical analysis and a direct numerical simulation is presented in section 2. Section 3 is devoted to a topological sensitivity analysis for the Quasi-Stokes equations. The obtained results are valid for a large class of cost functions. Finally, we illustrate the efficiency of the proposed method by a numerical test.

II. DIRECT SIMULATION

Let \( \Omega \) be a three dimensional flow domain representing the eutrophized lake. The used model is based on three dimensional Navier-Stokes equations for water flow in which we integrate the effect of momentum released by the injected bubbles by adding a local boundary condition for the velocity on the injector holes. In the presence of an injector \( \omega_{inj} \subset \Omega \), the velocity \( u(x,t) \) and the pressure \( p(x,t) \) solve the following system

\[
\begin{cases}
\frac{\partial u}{\partial t} + u \nabla u - \nu \Delta u + \nabla p = \mathcal{G} & \text{in } \Omega, \quad t \in [0,T] \\
\text{div } u = 0 & \text{in } \Omega, \quad t \in [0,T] \\
\rho u^0 = \rho u_0 & \text{in } \Gamma_0 \\
u u = u_d & \text{on } \Gamma \times [0,T]
\end{cases}
\] (1)

where \( \Omega = \Omega \backslash \omega_{inj} \) is the lake domain in the presence of the injector \( \omega_{inj} \), \( \nu \) is the water viscosity, \( \mathcal{G} \) is the gravitational force, \( T \) the final time of simulation, \( u_0 \) is the initial velocity field, \( \Gamma_i = \Gamma_s \cup \Gamma_w \cup \partial \omega_{inj} \) the boundary of \( \Omega \) and

\[
\begin{align*}
u_{inj} & = u_{wind} & \text{on } \Gamma_s : & \text{the surface lake boundary}, \\
u_{inj} & = 0 & \text{on } \Gamma_w : & \text{the bottom lake boundary}, \\
u_{inj} & = u_d & \text{on } \partial \omega_{inj} : & \text{the injector boundary}.
\end{align*}
\] (2)

Using characteristics method in (1), we obtain

\[
\begin{cases}
\alpha u^{n+1} + \nu \Delta u^{n+1} + \nabla p^{n+1} = F^{n+1} & \text{in } \Omega, \quad t \in [0,T] \\
\text{div } u^{n+1} = 0 & \text{in } \Omega, \quad t \in [0,T] \\
u^{n+1} = u_{inj} & \text{on } \Gamma_i, \quad t \in [0,T]
\end{cases}
\] (3)

where \( \alpha = \frac{1}{\Delta t} \), \( F^{n+1} = \frac{1}{\Delta t} u^n \alpha \chi^n + G \), \( u^{n+1} \) and \( p^{n+1} \) are the approximations of \( u \) and \( p \) on time \( t^{n+1} = (n+1)\Delta t \) and \( \chi^n(x) = X^n(t^{n+1}; x) \) represents the position at time \( t^{n+1} \) of the particle of fluid which is at point \( x \) at time \( t^n \).

Fig. 1. Aeration process

other hand, we look at the inverse problem: find the optimal injectors location generating the best motion in the fluid with respect to the aeration purpose. The optimal injectors location is characterized as the solution to a topological optimization problem. The topological sensitivity analysis is used to solve

M. Abdelwahed is with the Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Kingdom of Saudi Arabia (e-mail: mabdelwahed@ksu.edu.sa).
System (3) is solved iteratively for \( n = 0, 1, \ldots \). At each time step, we have to solve a steady state problem of Quasi-Stokes type having the following generic form.

For \( F \in L^2(\Omega_i)^3 \), and \( u_d \in H^1(\Gamma_i) \) such that \( \int_{\Gamma_i} u_d \cdot n \, ds = 0 \), find \( u \in H^1(\Omega_i)^3 \) and \( p \in L^2(\Omega_i) \) solutions of the problem

\[
\begin{align*}
   \alpha u - \nu \Delta u + \nabla p &= F & \text{in } \Omega_i \\
   \text{div } u &= 0 & \text{in } \Omega_i \\
   u &= u_d & \text{on } \Gamma_i.
\end{align*}
\]

Using 'P1 bubble/P1' mixed finite element method (see [3]) for the space approximation, we derive a linear matrix system. The resolution is based on Uzawa method and conjugate gradient algorithm [1].

For the numerical simulation, we used the following boundary conditions: \( u_{\text{wind}} = (0.01, 0, 0)m/s \) the wind velocity at the surface, no slip condition at the bottom and \( u_{\text{inj}} = (0, 0, 0.1m/s) \) the injection velocity on the injector. We present in figure 2 the numerical simulation of the aeration effect on the water flow in a three dimensional domain containing one injector obtained for \( T = 10mn.\)

\[
\begin{align*}
   \alpha u - \nu \Delta u + \nabla p &= F & \text{in } \bigcup_{k=1}^{m} \omega_{z_k,\varepsilon} \\
   \nabla u_{\varepsilon} &= 0 & \text{in } \bigcup_{k=1}^{m} \omega_{z_k,\varepsilon} \\
   u_{\varepsilon} &= u_d & \text{on } \Gamma \\
   u_{\varepsilon} &= u_{\text{inj}} & \text{on } \partial \omega_{z_k,\varepsilon}, 1 \leq k \leq m.
\end{align*}
\]

where \( u_{\text{inj}} \) is the injection velocity of the injector \( \omega_{z_k,\varepsilon}. \) Consider now a design function \( j \) having the form

\[
J_j(\Omega \setminus \bigcup_{k=1}^{m} \omega_{z_k,\varepsilon}) = J_j(u_{\varepsilon}),
\]

where \( J_j \) is a given cost function describing the optimization criteria and \( u_{\varepsilon} \) is the solution of (5). Our identification problem can be formulated as a topological optimization problem: find the optimal location of the injectors \( \omega_{z_k,\varepsilon} = z_k + \varepsilon \omega^k, 1 \leq k \leq m, \) inside the water reservoir domain \( \Omega \) minimizing the function \( j \).

\[
\begin{align*}
\text{Find } z^*_k & \in \Omega, 1 \leq k \leq m, \text{ such that :} \\
\{ \quad & \begin{align*}
   j(\Omega \setminus \bigcup_{k=1}^{m} \omega_{z_k,\varepsilon}) = \min_{\omega_{z_k,\varepsilon} \subset \Omega} j(\Omega \setminus \bigcup_{k=1}^{m} \omega_{z_k,\varepsilon}).
\end{align*}
\end{align*}
\]

To solve this optimization problem (P) we shall use the topological gradient method. It consists in studying the variation of the design function \( j \) with respect to a small topological perturbation of the domain \( \Omega. \)

A. Topological sensitivity analysis

In this section we derive a topological asymptotic expansion of the design function \( j \) with respect to the insertion of a small injector \( \omega_{z,\varepsilon} = z + \varepsilon \omega \) inside the domain \( \Omega. \) Next we assume that \( J_j \) satisfies the following assumptions.

\begin{align*}
\text{Hypothese 3.1: } & \quad i) \ J_j \text{ is differentiable with respect to } u, \\
& \quad ii) \text{ There exists a real number } \delta J \text{ such that } \forall \varepsilon \geq 0 \end{align*}

\[
J_j(u_{\varepsilon}) - J_j(u_0) = \delta J(u_{\varepsilon}) + \varepsilon \delta J + o(\varepsilon),
\]

where \( u_{\varepsilon} \) denotes the extension of \( u_{\varepsilon} \) in \( \Omega \) defined by \( u_{\varepsilon} = u_{\text{inj}} \) in \( \omega_{z,\varepsilon}. \)

We are now ready to derive the topological asymptotic expansion of the design function \( j \). It consists in computing the variation \( j(\Omega \setminus \omega_{z,\varepsilon}) - j(\Omega) \) when inserting a small injector inside the domain. The asymptotic expansion described in Theorem 3.1 is valid for arbitrary shaped holes and all cost function verifying the Hypothesis 3.1.
Theorem 3.1: If Hypothesis 3.1 holds, the function \( j \) has the following asymptotic expansion
\[
j(\Omega^{\infty}) - j(\Omega) = \varepsilon \left( - \int_{\partial \Omega} \eta(y) \ ds(y) \cdot v_0(z) + \delta J \right) + o(\varepsilon),
\]
where \( v_0 \) is the solution to the adjoint problem
\[
\begin{align*}
\alpha v_0 - \nu \Delta v_0 + \nabla q_0 &= - D J_0(u_0) \quad \text{in } \Omega, \\
\nabla v_0 &= 0 \quad \text{on } \Omega, \\
v_0 &= 0 \quad \text{on } \Gamma.
\end{align*}
\]
and \( \eta \in H^{-1/2}(\partial \Omega)^3 \) is the solution to the boundary integral equation.
\[
\int_{\partial \Omega} \varepsilon E(y-x) \eta(x) \ ds(x) = u_{inj} - u_0(z), \quad \forall y \in \partial \Omega. \quad (8)
\]
with \((E, \Pi)\) the fundamental solution of the Stokes equations
\[
E(y) = \frac{1}{8\pi \nu r} \left( I + e_y e_y^T \right), \quad \Pi(y) = \frac{y}{4\pi r^3},
\]
with \( r = ||y|| \), \( \epsilon_r = y/r \) and \( e_y^T \) is the transposed vector of \( e_y \).

Corollary 3.1: If \( \omega = B(0, 1) \), the density \( \eta \) is given explicitly:
\[
\eta(y) = - \frac{3\nu}{2} u_0(z) \quad \forall y \in \partial \Omega \quad \text{and under the hypothesis of theorem 3.1},
\]
we have
\[
j(\Omega^{\infty}) - j(\Omega) = \varepsilon \left( 6\pi \nu u_0(z) \cdot v_0(z) + \delta J \right) + o(\varepsilon).
\]

B. Numerical results
Aeration is considered as the best remedial action against eutrophication. This process consists in inserting someinjector holes \( \omega_k \) in the bottom layer of the reservoir in order to create a dynamic and aerate the water. We suppose that a "good" waterreservoir aeration can be described by a target velocity \( U_g \). This solution isobtained by the dynamic aeration process using more than 1000 injectors located at the bottom layer \( \Omega_b \). We aim to findthe optimal location of a fixed number of injectors \( m \) in order to approximate the wanted flow \( U_g \).

For this numerical test, we consider in figure 4 a constructed solution representing the velocity field \( U_g \). This solution isobtained by the dynamic aeration process using more than 1000 injectors located at the bottom layer \( \Omega_b \). We aim to findthe optimal location of a fixed number of injectors \( m \) in order to approximate the wanted flow \( U_g \).

Using our algorithm with 25 injectors (i.e. \( m = 25 \)), we show in figure 7 the obtained flow during the optimization processat iterations 1, 3 and 5. The optimal injectors location is given...
in figure 6. Figure 5 shows a vertical cut of the wanted and obtained flows in the measurement domain \( \Omega_m \). We remark that we obtain approximately the same flow.

![Fig. 6. Injectors location obtained during the optimization process: lateral view (left) and top view (right)](image)

![Fig. 7. Velocities field obtained during the optimization process.](image)

**REFERENCES**


