I-Vague Normal Groups

Zelalem Teshome Wale

Abstract—The notions of I-vague normal groups with membership and non-membership functions taking values in an involutory dually residuated lattice ordered semigroup are introduced which generalize the notions with truth values in a Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval [0, 1]. Various operations and properties are established.

Keywords—Involutory dually residuated lattice ordered semigroup, I-vague set, I-vague group and I-vague normal group.

I. INTRODUCTION

VAGUE groups are studied by M. Demirci[2]. R. Biswas[1] defined the notion of vague groups analogous to the idea of Rosenfeld [4]. He defined vague normal groups of a group and studied their properties. N. Ramakrishna[3] studied vague normal groups and introduced vague normalizer and vague centralizer.

In his paper, T. Zelalem [9] studied the concept of I-vague groups. In this paper using the definition of I-vague groups, we defined and studied I-vague normal groups where I is an involutory DRL-semigroup. To be self contained we shall recall some basic results in [5], [6], [7], [9] in this paper.

II. DUALLY RESIDUATED LATTICE ORDERED SEMIGROUP

Definition 2.1: [5] A system $A = (A, +, 0, ≤, −)$ is called a dually residuated lattice ordered semigroup with I in short DRL-semigroup if and only if

i) $A = (A, +)$ is a commutative semigroup with zero $0$;

ii) $A = (A, ≤)$ is a lattice such that $a + (b ∪ c) = (a + b) ∪ (a + c)$ and $a + (b ∩ c) = (a + b) ∩ (a + c)$ for all $a, b, c ∈ A$;

iii) Given $a, b ∈ A$, there exists a least $x$ in $A$ such that $b + x ≥ a$, and we denote this $x$ by $a - b$ (for a given $a, b$ this $x$ is uniquely determined);

iv) $(a - b) ∪ 0 + b ≤ a ∪ b$ for all $a, b ∈ A$;

v) $a - a ≥ 0$ for all $a ∈ A$.

Theorem 2.2: [5] Any DRL-semigroup is a distributive lattice.

Definition 2.3: [10] A DRL-semigroup A is said to be involutory if there is an element $1(≠ 0)$ is the identity w.r.t. $+$ such that

i) $a + (1 - a) = 1 + 1$;

ii) $1 - (1 - a) = a$ for all $a ∈ A$.

Theorem 2.4: [6] In a DRL-semigroup with 1, $1$ is unique.

Theorem 2.5: [6] If a DRL-semigroup contains a least element $x$, then $x = 0$. Dually, if a DRL-semigroup with 1 contains a largest element $a$, then $a = 1$.

Zelalem Teshome: Department of Mathematics, Addis Ababa University, Addis Ababa, Ethiopia.

E-mail: zelalamwale@yahoo.com or zelalem.wale@gmail.com

Throughout this paper let $I = (I, +, −, ∨, ∩, 0, 1)$ be a dually residuated lattice ordered semigroup satisfying $1 - (1 - a) = a$ for all $a ∈ I$.

Lemma 2.6: [10] Let $I$ be the largest element of $I$. Then for $a, b ∈ I$

(i) $a + (1 - a) = 1$

(ii) $1 - a = 1 - b ⇐⇒ a = b.$

(iii) $1 - (a ∪ b) = (1 - a) ∩ (1 - b)$.

Lemma 2.7: [10] Let $I$ be complete. If $a_α ∈ I$ for every $α ∈ Δ$, then

(i) $1 - ∨ α a_α = ∩ α (1 - a_α)$.

(ii) $1 - ∩ α a_α = ∪ α (1 - a_α)$.

III. I-VAGUE SETS

Definition 3.1: [10] An I-vague set $A$ of a non-empty set $G$ is a pair $(t_A, f_A)$ where $t_A: G → I$ and $f_A: G → I$ with $t_A(x) ≤ 1 - f_A(x)$ for all $x ∈ G$.

Definition 3.2: [10] The interval $[t_A(x), 1 - f_A(x)]$ is called the I-vague value of $x ∈ G$ and is denoted by $V_A(x)$.

Definition 3.3: [10] Let $B_1 = [a_1, b_1]$ and $B_2 = [a_2, b_2]$ be two I-vague values. We say $B_1 ≥ B_2$ if and only if $a_1 ≥ a_2$ and $b_1 ≥ b_2$.

Definition 3.4: [10] An I-vague set $A = (t_A, f_A)$ of $G$ is said to be contained in an I-vague set $B = (t_B, f_B)$ of $G$ written as $A ⊆ B$ if and only if $t_A(x) ≤ t_B(x)$ and $f_A(x) ≥ f_B(x)$ for all $x ∈ G$. A is said to be equal to B written as $A ≡ B$ if and only if $A ⊆ B$ and $B ⊆ A$.

Definition 3.5: [10] An I-vague set $A$ of $G$ with $V_A(x) = V_A(y)$ for all $x, y ∈ G$ is called a constant I-vague set of $G$.

Definition 3.6: [10] Let $A$ be an I-vague set of a non-empty set $G$. Let $A_{α, β} = \{x ∈ G : V_A(x) ≥ [α, β]\}$ where $α, β ∈ I$ and $α ≤ β$. Then $A_{α, β}$ is called the $(α, β)$ cut of the I-vague set $A$.

Definition 3.7: Let $S ⊆ G$. The characteristic function of $S$ denoted as $χ_S = (t_{χ_S}, f_{χ_S})$, which takes values in $I$ is defined as follows:

$t_{χ_S}(x) = \begin{cases} 1 & \text{if } x ∈ S \\ 0 & \text{otherwise} \end{cases}$

and

$f_{χ_S}(x) = \begin{cases} 0 & \text{if } x ∈ S \\ 1 & \text{otherwise.} \end{cases}$

$χ_S$ is called the I-vague characteristic set of $S$ in $I$. Thus

$V_{χ_S}(x) = \begin{cases} [1, 1] & \text{if } x ∈ S; \\ [0, 0] & \text{otherwise.} \end{cases}$

Definition 3.8: [10] Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be I-vague sets of a set $G$.

(i) Their union $A ∪ B$ is defined as $A ∪ B = (t_{A∪B}, f_{A∪B})$ where $t_{A∪B}(x) = t_A(x) ∨ t_B(x)$ and
(ii) Their intersection $A \cap B$ is defined as $A \cap B = (t_{A \cap B}, f_{A \cap B})$ where $t_{A \cap B}(x) = t_A(x) \wedge t_B(x)$ and $f_{A \cap B}(x) = f_A(x) \vee f_B(x)$ for each $x \in G$.

**Definition 3.9:** [10] Let $B_1 = [a_1, b_1]$ and $B_2 = [a_2, b_2]$ be I-vague values. Then

(i) $\text{isup}(B_1, B_2) = [\sup\{a_1, a_2\}, \sup\{b_1, b_2\}]$.

(ii) $\text{infs}(B_1, B_2) = [\inf\{a_1, a_2\}, \inf\{b_1, b_2\}]$.

**Lemma 3.10:** [10] Let $A$ and $B$ be I-vague sets of a set $G$. Then $A \cup B$ and $A \cap B$ are also I-vague sets of $G$.

Let $x \in G$. From the definition of $A \cup B$ and $A \cap B$ we have

(i) $V_{A \cup B}(x) = \text{isup}\{V_A(x), V_B(x)\}$;

(ii) $V_{A \cap B}(x) = \text{infs}\{V_A(x), V_B(x)\}$.

**Definition 3.11:** [10] Let $\{A_i : i \in \Delta\}$ be a non empty family of I-vague sets of $G$ where

$A_i = (t_{A_i}, f_{A_i})$. Then

(i) $\bigcap_{i \in \Delta} A_i = (\bigwedge_{i \in \Delta} t_{A_i}, \bigvee_{i \in \Delta} f_{A_i})$

(ii) $\bigcup_{i \in \Delta} A_i = (\bigvee_{i \in \Delta} t_{A_i}, \bigwedge_{i \in \Delta} f_{A_i})$

**Lemma 4.12:** [9] Let $A$ be an I-vague group of a group $G$. Then $A \subseteq G$ is an I-vague normal group of $G$ if and only if for all $\alpha, \beta \in G$ and $\gamma \leq \delta \leq \beta$

is an I-vague normal group of $G$.

**Theorem 4.10:** [9] An I-vague set $A$ of a group $G$ is an I-vague group of $G$ if and only if for all $\alpha, \beta \in G$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha, \beta)}$ is a subgroup of $G$ whenever it is non empty.

**Theorem 4.11:** [9] Let $A$ be an I-vague group of a group $G$. If $V_A(xy^{-1}) = V_A(x)$ for $y, x \in G$, then $V_A(y) = V_A(y)$. Let $A$ be an I-vague group of a group $G$. Then $G_A = \{x \in G : V_A(x) = V_A(e)\}$ is a subgroup of $G$.

**V. I-VAGUE NORMAL GROUPS**

**Definition 5.1:** Let $G$ be a group. An I-vague group $A$ of a group $G$ is called an I-vague normal group of $G$ if for all $x, y \in G, V_A(xy^{-1}) = V_A(y)$. Let $G$ be an I-vague normal group of $G$. If $G$ is abelian, then every I-vague group of $G$ is an I-vague normal group of $G$.

**Lemma 5.2:** Let $A$ be an I-vague group of a group $G$. Suppose that $A$ is an I-vague normal group of $G$. Let $x, y \in G$. Then $V_A(xy^{-1}) = V_A(yx^{-1})$. Thus $V_A(x) = V_A(yx^{-1})$. Conversely, suppose that $V_A(x) = V_A(yx^{-1})$ for all $x, y \in G$. Then $V_A(xy) = V_A(xy^{-1}) = V_A(yx)$. We have $V_A(xy) = V_A(yx)$. Hence the lemma follows.

**Lemma 5.3:** Let $H$ be a normal subgroup of $G$ and $[\gamma, \delta] \leq [\alpha, \beta]$ for $\alpha, \beta, \gamma, \delta \in I$ with $\alpha \leq \beta$ and $\gamma \leq \delta$. Then the I-vague set $A$ of $G$ defined by

$A(x) = \begin{cases} [\alpha, \beta] & \text{if } x \in H \\
[\gamma, \delta] & \text{otherwise} \end{cases}$

is an I-vague normal group of $G$.

**Proof:** Let $H$ be a normal subgroup of $G$. By Lemma 4.5, $A$ is an I-vague group of $G$. We show that $V_A(xy^{-1}) = V_A(xy^{-1})$ for every $x, y \in G$. Let $x, y \in G$. If $x \in H$, then $xy^{-1} \in H$. Thus $V_A(xy^{-1}) = V_A(xy^{-1})$. If $x \notin H$, then $xy^{-1} \notin H$. Thus $V_A(xy^{-1}) = V_A(xy^{-1})$. Hence $V_A(xy) = V_A(xy^{-1})$ for every $x, y \in G$. Therefore $A$ is an I-vague normal group of $G$.

**Lemma 5.4:** Let $H \neq \emptyset$. The I-vague characteristic set of $H$, $\chi_n$ is an I-vague normal group of $G$ if $H$ is a normal subgroup of $G$.

**Proof:** Suppose that $H$ is a normal subgroup of $G$. By Lemma 5.3, $\chi_n$ is an I-vague normal group of $G$ since

$V_{\chi_n}(x) = \begin{cases} [1, 1] & \text{if } x \in H \\
[0, 0] & \text{otherwise} \end{cases}$

Conversely, suppose that $\chi_n$ is an I-vague normal group of $G$. We show that $H$ is a normal subgroup of $G$. By Lemma 4.6, $H$ is a subgroup of $G$. Let $y \in H$ and $x \in G$. Therefore $\chi_n$ is an I-vague normal group of $G$.
Remark

Proof: of $G$ as we have seen in I-vague groups [9].

Let us define $A \in \mathcal{B}$ be a constant I-vague group of $G$. Then $A \subseteq V$ by lemma 4.8.

Theorem 5.5: Let $A = \bigcap_{i \in \Delta} A_i$. Then $A$ is an I-vague normal group of $G$.

Proof: Let $A$ be an I-vague normal group of $G$ and $B$ be a constant I-vague group of $G$. Then $A \subseteq B$ is an I-vague normal group of $G$.

Theorem 5.6: Let $A$ be an I-vague normal group of $G$ and $B$ be a constant I-vague group of $G$. Then $A \subseteq B$ is an I-vague normal group of $G$.

Proof: Let $A$ be an I-vague normal group of $G$ and $B$ be a constant I-vague group of $G$. Hence $A \subseteq B$ is an I-vague normal group of $G$.

Theorem 5.7: Let $A$ be an I-vague normal group of $G$.

Proof: Let $A$ be an I-vague normal group of $G$ as we have seen in I-vague groups [9].

Theorem 5.8: An I-vague set $A$ of a group $G$ is an I-vague normal group of $G$ if and only if for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha, \beta)}$ is a normal subgroup of $G$ whenever it is non-empty.

Proof: By theorems 4.10, an I-vague set $A$ of a group $G$ is an I-vague normal group of $G$ if and only if for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha, \beta)}$ is a normal subgroup of $G$ whenever it is non-empty.

Suppose that $A$ is an I-vague normal group of $G$. Consider $A_{(\alpha, \beta)}$. Let $y \in A_{(\alpha, \beta)}$ and $x \in G$. We prove that $x y x^{-1} \in A_{(\alpha, \beta)}$.

Now we prove that $x y x^{-1} \in H$.

Proof: We prove that $A_{(\alpha, \beta)}$ is a normal subgroup of $G$.

Hence $A_{(\alpha, \beta)}$ is a normal subgroup of $G$.

Hence the lemma holds true.

Theorem 5.9: If $A$ and $B$ are I-vague normal groups of $G$, then $A \subseteq B$ is also an I-vague normal group of $G$.

Proof: We prove that $A_{(\alpha, \beta)}$ is a normal subgroup of $G$ and $B$ is a constant I-vague group of $G$. Then $A \subseteq B$ is an I-vague normal group of $G$.

Hence $A_{(\alpha, \beta)}$ is a normal subgroup of $G$.

Hence the theorem follows.

Theorem 5.10: If $A$ is an I-vague normal group of $G$, then $G_A$ is a normal subgroup of $G$.

Proof: We prove that $G_A$ is a normal subgroup of $G$.

By lemma 4.12, $G_A = \{x \in G : V_A(x) = V_A(e)\}$ is a subgroup of $G$. Now we show that $x y x^{-1} \in G_A$ for all $x, y \in G_A$.

Thus $G_A$ is a normal subgroup of $G$.

Theorem 5.11: Let $A$ be an I-vague group of $G$. Then $A$ is an I-vague normal group if $G$.

Proof: We prove that $A_{(\alpha, \beta)}$ is a normal subgroup of $G$ and $B$ is an I-vague normal group of $G$.

Hence $A_{(\alpha, \beta)}$ is a normal subgroup of $G$.

Hence $A_{(\alpha, \beta)}$ is a normal subgroup of $G$.

Hence $A_{(\alpha, \beta)}$ is a normal subgroup of $G$. 

Conversely, suppose that for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the non empty set $A_{(\alpha, \beta)}$ is a normal subgroup of $G$.

Now it remains to prove that $V_A(y) = V_A(x y^{-1})$ for all $x, y \in G$.

Hence the theorem follows.

Theorem 5.12: If $A$ is an I-vague normal group of $G$, then $G_A$ is a normal subgroup of $G$.

Proof: We prove that $G_A$ is a normal subgroup of $G$.

By lemma 4.12, $G_A = \{x \in G : V_A(x) = V_A(e)\}$ is a subgroup of $G$. Now we show that $x y x^{-1} \in G_A$ for all $x, y \in G_A$.

Thus $G_A$ is a normal subgroup of $G$.

Hence the theorem follows.
We have

By definition, \( A \) is an I-vague normal group of \( G \).

Conversely, assume that \( N(A) = G \). For all \( y \),

Let \( N \). Then \( a^{-1} \in N(A) \) and \( ab \in N(A) \).

Let \( a \in N(A) \). Then \( V_a(a^{-1}x) = V_a(x) \) for all \( x \in G \).

Hence \( V_a(a^{-1}x) = V_a(x) \), so \( a^{-1} \in N(A) \).

Let \( a, b \in N(A) \). Then

Then \( ab \in N(A) \). Therefore \( N(A) \) is a subgroup of \( G \).

(iii) Suppose that \( A \) is an I-vague normal group of \( G \).

We prove that \( N(A) = G \).

Let \( a \in G \). Since \( A \) is an I-vague normal group of \( G \),

\( V_a(a^{-1}x) = V_a(x) \) for all \( x \in G \). It follows that \( a \in N(A) \).

Hence \( G \subseteq N(A) \).

Since \( N(A) \subseteq G \), \( G = N(A) \).

Conversely, assume that \( N(A) = G \). For all \( a \), \( x \in G \),

\( V_a(a^{-1}x) = V_a(x) \).

By definition, \( A \) is an I-vague normal group of \( G \).

Therefore \( V_a(x) = V_a(a^{-1}x) \).

Hence \( V_a(x) = V_a(a^{-1}x) \) for all \( x \in G \) and \( x \in G \).

For \( y \in G \), \( V_a(xy^{-1}) \geq \inf \{ V_a(x), V_a(yx^{-1}) \} \).

Hence \( V_a(xy^{-1}) \geq V_a(y) \) for \( y \in G \) and \( x \in G \).

Thus \( V_a(xy^{-1}) \geq V_a(y) \) where \( x \in G \) and \( y \in G \). Put \( xy^{-1} \) instead of \( y \).

We have \( V_a(xy^{-1}x) \geq V_a(yx^{-1}) \) and hence \( V_a(y) \geq V_a(xy^{-1}) \).

Therefore \( V_a(y) = V_a(xy^{-1}) \) for each \( y \in G \).

Thus \( x \in N(A) \). Therefore \( GVA \subseteq N(A) \).

Since \( GVA \) is a subgroup of \( G \) and \( GVA \subseteq N(A) \), \( GVA \) is a subgroup of \( N(A) \).

Now we show that \( ygy^{-1} \in GVA \) for all \( a \in GVA \) and for all \( y \in N(A) \).

Since \( y \in N(A) \), \( V_a(ygy^{-1}) = V_a(a) \). Since \( a \in GVA \),

\( V_a(a) = V_a(e) \). Hence \( V_a(ygy^{-1}) = V_a(e) \), so \( ygy^{-1} \in GVA \). Therefore \( GVA \) is a normal subgroup of \( N(A) \).

**Definition 5.15:** Let \( A \) be an I-vague group of a group \( G \). Then the set

\( C(A) = \{ a \in G : V_A([a,x]) = V_A(e) \} \) for all \( x \in G \) is called an I-vague centralizer of \( A \).

**Theorem 5.16:** Let \( A \) be an I-vague group of a group \( G \). Then \( C(A) \) is a normal subgroup of \( G \).

**Proof:** Let \( A \) be an I-vague group of \( G \). We prove that \( C(A) = \{ a \in G : V_A([a,x]) = V_A(e) \} \) for all \( x \in G \) is a normal subgroup of \( G \).

Step(1) We show that \( a \in C(A) \) implies \( V_A(ax) = V_A(ax) \) for all \( x \in G \).

Let \( a \in C(A) \). Then \( V_A([a,x]) = V_A(e) \) for all \( x \in G \).

\( V_A([a,x]) = V_A(e) \Rightarrow V_A(a^{-1}xax) = V_A(e) \)

\( \Rightarrow V_A(a^{-1}xax) = V_A(e) \)

\( \Rightarrow V_A(a^{-1}xax) = V_A(e) \) by thm 4.11

\( \Rightarrow V_A(ax) = V_A(ax) \).

Therefore \( V_A(ax) = V_A(ax) \) for all \( x \in G \).

Step(2) We show that \( a \in C(A) \) implies \( V_A([x,a]) = V_A(e) \) for all \( x \in G \).

\( V_A([x,a]) = V_A(e) \) by step(1)

\( V_A((x^{-1}ax)a^{-1}) = V_A((x^{-1}ax)a^{-1}) \)

\( V_A((x^{-1}ax)a^{-1}) = V_A((x^{-1}ax)a^{-1}) \)

\( V_A([a,x]) = V_A(e) \) for each \( a \in C(A) \) and for all \( x \in G \).

Step(3) We prove that \( C(A) \) is a subgroup of \( G \).

We show that (i) \( a \in C(A) \) implies \( a^{-1} \in C(A) \).

(ii) \( a, b \in C(A) \) implies \( ab \in C(A) \).

Now proof of (i)

For all \( x \in G \), \( V_A([a^{-1},x]) = V_A(ax^{-1}a^{-1}x) \)

\( = V_A(x^{-1}ax) \) by step(1)

\( = V_A(x^{-1}ax) \)

\( = V_A([a,x]) = V_A(e) \).
\[ V_A(g^{-1}a^{-1}gaa^{-1}x^{-1}g^{-1}agx) = V_A(g, a)g^{-1}(gx)^{-1}agx \]
\[ = V_A([g, a][a, gx]) \geq \inf\{V_A([g, a]), V_A([a, gx])\} = \inf\{V_A(e), V_A(e)\} = V_A(e). \]

Hence \( V_A([g^{-1}ag, x]) \geq V_A(e). \)

Since \( V_A(e) \geq V_A([g^{-1}ag, x]) \), \( V_A([g^{-1}ag, x]) = V_A(e) \).

This implies \( g^{-1}ag \in C(A) \).

From step(3) and step(4), we have \( C(A) \) is a normal subgroup of \( G \).

**Theorem 5.17:** Let \( A \) be an I-vague normal group of a group \( G \). Then \( GV_A \) is a subgroup of \( C(A) \).

**Proof:** Let \( A \) be an I-vague normal group of a group \( G \). We prove that \( GV_A \) is a subgroup of \( C(A) \).

Let \( x \in GV_A \). Then \( V_A(x) = V_A(e) \). Consider \( V_A([x, y]) \) for each \( y \in G \).

\[ V_A([x, y]) = V_A(x^{-1}(y^{-1}xy)) \geq \inf\{V_A(x^{-1}), V_A(y^{-1}xy)\} \]
\[ = \inf\{V_A(x), V_A(x)\} \]
\[ = V_A(x) = V_A(e). \]

Hence \( V_A([x, y]) \geq V_A(e) \).

Since \( V_A(e) \geq V_A([x, y]) \), \( V_A([x, y]) = V_A(e) \).

By the definition of \( C(A) \), \( x \in C(A) \).

Thus \( GV_A \subseteq C(A) \). Since \( GV_A \) is a subgroup of \( G \), \( GV_A \) is a subgroup of \( C(A) \).

**ACKNOWLEDGMENT**

The author would like to thank Prof. K. L. N. Swamy and Prof. P. Ranga Rao for their valuable suggestions and discussions on this work.

**REFERENCES**


