Exact Solutions of the Helmholtz equation via the Nikiforov-Uvarov Method

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***Abstract*—**The Helmholtz equation often arises in the study of physical problems involving partial differential equation. Many researchers have proposed numerous methods to find the analytic or approximate solutions for the proposed problems. In this work, the exact analytical solutions of the Helmholtz equation in spherical polar coordinates are presented using the Nikiforov-Uvarov (NU) method. It is found that the solution of the angular eigenfunction can be expressed by the associated-Legendre polynomial and radial eigenfunctions are obtained in terms of the Laguerre polynomials. The special case for *k=0*, which corresponds to the Laplace equation is also presented.

***Keywords*—**Helmholtz equation, Nikiforov-Uvarov method, exact solutions, eigenfunctions.

I. Introduction

THE Helmholtz equation is used to describe many mathematical and physical applications including electromagnetics, wave propagation [1], heat conduction [2], acoustic cavity problem [3] and many others. It has also been used to describe the vibration of a structure [4] and the scattering of a wave [5]. Many different techniques such as the finite difference method [6], the homotopy perturbation method [7], [8] and the variational iteration method [9] have been introduced to solve the Helmholtz equation numerically and analytically. The NU method has been introduced for solving the hypergeometric type second-order differential equations [10] appeared in the time-independent problems. Recently, this method has been used to solve Schrödinger equation for some well known potentials [11]–[16], Dirac, Klein–Gordon equations for Coulomb potential [17], [18] and some physical potentials such as Woods-Saxon potential [19], Poschl-Teller potentials [20], Hulthen potentials [21], the Manning-Rosen potential [22], and the Eckart potential [23]-[25].

In the present work, our main objective is to solve Helmholtz equation in spherical polar coordinates by the NU method.

This paper is organized as follows: In Section II, we briefly introduce the NU method. We consider the separation of variables to obtain the eigenfunctions of the Helmholtz equation in spherical polar coordinates in Section III. Finally, conclusions are drawn in Section IV.

II. Nikiforov-Uvarov Method

The NU Method [26] is based on the solutions of general second-order linear differential equation with special functions. It has been extensively used to solve the non relativistic Schrödinger equation and Schrödinger-like equations. This method in general reduces the second-order linear differential equation to the following form

$$\frac{d^2\psi(s)}{ds^2}+\frac{\bar{\tau}(s)}{\sigma(s)}\frac{d\psi(s)}{ds}+\frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{1}$$

where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials, at most second degree, and $\bar{\tau}(s)$ is a polynomial, at most first degree. To find the particular solution of (1), one can use the following form

$$\psi_n(s)=\phi(s)y_n(s) \tag{2}$$

It reduces to an equation of hypergeometric type

$$\sigma(s)\frac{d^2y_n(s)}{ds^2}+\tau(s)\frac{dy_n(s)}{ds}+\lambda y_n(s)=0 \tag{3}$$

where λ is a constant given in the form

$$\lambda=\lambda_n=-n\tau'(s)-\frac{n(n-1)}{2}\sigma''(s),\quad n=1,2,3,... \tag{4}$$

$\phi(s)$ is defined as logarithmic derivative

$$\frac{\phi'(s)}{\phi(s)}=\frac{\pi(s)}{\sigma(s)} \tag{5}$$

$y_n(s)$ is the hypergeometric-type function whose polynomials solutions are given by Rodrigues relation

$$y_n(s)=\frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right] \tag{6}$$

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where B_n is the normalization constant and $\rho(s)$ is the weight function satisfying

$$\frac{d}{ds}\left[\sigma(s)\rho(s)\right]=\tau(s)\rho(s) \tag{7}$$

where $\tau(s)=\bar{\tau}(s)+2\pi(s)$ satisfies the condition $\tau'(s)<0$ [26].

The function $\pi(s)$ and the parameter λ, required for this method, are defined as follows:

$$\pi(s)=\frac{\sigma'-\bar{\tau}}{2}\pm\sqrt{\left(\frac{\sigma'-\bar{\tau}}{2}\right)^2-\bar{\sigma}+\kappa\sigma} \tag{8}$$

$$\lambda=\kappa+\pi'(s) \tag{9}$$

In order to find the value of κ, the expression under the square root must be square of polynomial

III. Helmholtz Equation

Let us consider the Helmholtz equation,

$$\nabla^2 U+k^2U=0 \tag{10}$$

which represents the time-independent linear partial differential equation, results from applying the technique of separation of variables to reduce the complexity of the analysis. The interpretation of the unknown U depends on what the equation models. The most common areas are wave propagation problems and quantum mechanics, in which case U is the amplitude of a time-harmonic wave and the orbitals for an energy state, respectively.

Here ∇^2 is the Laplacian and k is the wavenumber. In spherical polar coordinates, (1) can be written

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}+k^2U=0 \tag{11}$$

We seek the solutions to (11) in the form

$$U=R(r)T(\theta)\Phi(\varphi) \tag{12}$$

Substituting (12) into (11) and dividing by U gives

$$\frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}+\left[k^2-\frac{\ell(\ell+1)}{r^2}\right]R(r)=0 \tag{13}$$

$$\frac{d^2T(\theta)}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{dT(\theta)}{d\theta}+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]T(\theta)=0 \tag{14}$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi)=0 \tag{15}$$

where m^2 and $\ell(\ell+1)$ are separation constants. The solution in (15) is periodic and must satisfy the periodic boundary condition $\Phi(\varphi+2\pi)=\Phi(\varphi)$, from which we obtain

$$\Phi_m(\varphi)=\frac{1}{\sqrt{2\pi}}e^{im\varphi} \qquad m=0,\pm1,\pm2,... \tag{16}$$

Furthermore, (13) and (14) are radial and polar angle equations respectively and may be solved using the Nikiforov-Uvarov Method in the following sections.

A. Solutions of Polar Angle Equation

In order to apply the NU Method; we introduce a new variable $s=\cos\theta$. Hence (14) becomes

$$\frac{d^2T(s)}{ds^2}-\frac{2s}{1-s^2}\frac{dT(s)}{ds}+\frac{1}{(1-s^2)^2}\left[\ell(\ell+1)(1-s^2)-m^2\right]T(s)=0 \tag{17}$$

To solve (17) by means of the NU method, we should compare it with (1), we get

$$\bar{\tau}(s)=-2s\,,\sigma(s)=1-s^2,\ \bar{\sigma}(s)=\ell(\ell+1)(1-s^2)-m^2 \tag{18}$$

Substituting the above expressions into relation (8) leads to

$$\pi(s)=\pm\sqrt{(\ell(\ell+1)-\kappa)s^2-\ell(\ell+1)+\kappa+m^2} \tag{19}$$

According to the NU method, the constant parameter κ can be determined from the condition that the expression under the square root must be the square of a polynomial of first degree, so we have the following four possible values:

$$\pi(s)=\pm ms \qquad \text{for } \kappa_1=m^2+\ell(\ell+1) \tag{20}$$

$$\pi(s)=\pm m \qquad \text{for } \kappa_2=\ell(\ell+1) \tag{21}$$

After proper choice of the polynomial $\pi(s)$ and κ, we can write the function $\tau(s)$ which has a negative derivative as follows:

$$\tau(s)=-2(m+1)s;\ \tau'(s)=-2(m+1)<0.\quad \text{for } \kappa_1 \tag{22}$$

According to (4) and (9) the values n can be obtained $n=\ell-m$

From (5) and (7), we obtain

$$\phi(s)=(1-s^2)^{m/2} \tag{23}$$

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$$\rho(s) = (1-s^2)^m \tag{24}$$

Inserting (24) into (6), $y_n(s)$ can be found as follows:

$$y_n(s) = B_n(1-s^2)^{-m}\frac{d^n}{ds^n}\left[(1-s^2)^{n+m}\right] \tag{25}$$

The polar part of the (12) is found to be

$$T_n(\theta) = C_n(1-s^2)^{-m/2}\frac{d^{\ell-m}}{ds^{\ell-m}}\left[(1-s^2)^{\ell}\right] \tag{26}$$

where C_n is the normalization constant

$$C_n = \sqrt{\frac{(2n+2m+1)n!}{2(n+2m)!}} \tag{27}$$

and $T_n(\theta)$ is for the associated-Legendre function $P_\ell^m(s)$. Thus the angle part solution of the Helmholtz equation can be written as

$$T(\theta) = \sqrt{\frac{(2n+2m+1)n!}{2(n+2m)!}}P_\ell^m(\cos\theta) \tag{28}$$

B. Solutions of Radial Equation

We now focus on the radial eigenfunction of the Helmholtz equation, with change variable $r = s$. Equation (13) takes the form

$$\frac{d^2R(s)}{ds^2}+\frac{2}{s}\frac{dR(s)}{ds}+\frac{1}{s^2}\left[k^2s^2-\ell(\ell+1)\right]R(s)=0 \tag{29}$$

By comparing (29) with (1) we have

$$\bar{\tau}(s) = 2,\ \sigma(s) = s,\ \bar{\sigma}(s) = k^2s^2 - \ell(\ell+1) \tag{30}$$

Using (8), $\pi(s)$ can be written as

$$\pi(s) = -\frac{1}{2}\pm\left[iks+\frac{1}{2}\sqrt{1+4\ell(\ell+1)}\right]\ \textit{for}\ \kappa_1 = ik\sqrt{1+4\ell(\ell+1)} \tag{31}$$

$$\pi(s) = -\frac{1}{2}\pm\left[-iks+\frac{1}{2}\sqrt{1+4\ell(\ell+1)}\right]\ \textit{for}\ \kappa_2 = -ik\sqrt{1+4\ell(\ell+1)} \tag{32}$$

Making the appropriate choice for the polynomials as

$$\pi(s) = -\frac{1}{2}+\left[-iks+\frac{1}{2}\sqrt{1+4\ell(\ell+1)}\right] \tag{33}$$

gives the $\tau(s)$ function

$$\tau(s) = 1+\left[-2iks+\sqrt{1+4\ell(\ell+1)}\right] \tag{34}$$

$$\tau'(s) = -2iks < 0 \tag{35}$$

The corresponding weight function $\rho(s)$ has the form

$$\rho(s) = e^{-2iks}s^{1+2\ell} \tag{36}$$

Using the condition (5) $\phi(s)$ is given by

$$\phi(s) = e^{-iks}s^{\ell} \tag{37}$$

By substituting (36) into the Rodrigues relation (6) we obtain

$$y_n(s) = C_nL_n^{2\ell+1}(2i\kappa s) \tag{38}$$

where $L_n(s)$ being the Laguerre polynomials and C_n is the normalization constant.

The radial part of the eigenfunction of the Helmholtz equation has the form

$$R(s) = Ns^{\ell}e^{-iks}L_n^{2\ell+1}(2i\kappa s) \tag{39}$$

with N is a new normalization factor.

The associated Laplace equation solutions are obtained by setting $k = 0$ in (29). The $T(s)$ solution is unchanged but the solution (39) becomes

$$R(s) = Ns^{\ell} \tag{40}$$

The total eigenfunction of the Helmholtz equation can be obtained from the combined solutions of (16), (28) and (39) as

$$U(s,\theta,\phi) = N\sqrt{\frac{(2n+2m+1)n!}{4\pi(n+2m)!}}s^{\ell}e^{-i(ks-m\phi)}.P_\ell^m(\cos\theta)L_n^{2\ell+1}(2i\kappa s) \tag{41}$$

IV. Conclusion

We have obtained the exact solutions of the Helmholtz equation using the NU method. We have found that the angular and radial eigenfunctions can be expressed in terms of the associated-Legendre and Laguerre polynomials respectively. In the case of $k = 0$ (Laplace equation), the radial solutions are given by (40). It may be concluded that the NU method is simple and powerful in finding the analytical solutions for a wide class of such problems.

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REFERENCES

[1] M. Kawahara and K. Kashiyama, Boundary type finite element method for surface wave motion based on the trigonometric function interpolation Int. J. Numer. Meth. Eng. 21, pp. 1833-1852, 1985.

[2] R. A. Altenkirch, M Rezayat, R. Eichhorn, and F. J. Rizzo A study of heat conduction forward of flame in solids, by the boundary integral equation method," Trans. ASME Ser. C. J. Heat Transfer 104, pp. 734-740, 1982.

[3] J T Chen, F. C .Wong Dual formulation of multiple reciprocity method for the acoustic mode of a cavity with a thin partition. J. Sound Vibration 217**,** 75–95, 1998.

[4] Beskos DE Boundary element method in dynamic analysis: Part II (1986–1996). ASME Appl. Mech. Rev. 50, pp**.** 149–197, 1997.

[5] W. S Hall, X. Q. Mao A boundary element investigation of irregular frequencies in electromagnetic scattering. Eng. Anal. Bound. Elem*.* 16**,** pp**.** 245–252, 1995.

[6] S.M. El-Sayed and D.Kaya, Comparing numerical methods for Helmholtz equation model problem, Appl. Math. Comp.150**,** pp**.** 763-773, 2004.

[7] D.D. Ganji, A. Rajabi, Assessment of homotopy-perturbation and Perturbation methods in heat radiation equations, International. Commun. Heat Mass Transfer 33 (3) pp. 391–400, 2006.

[8] J.H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Eng. 178**,** pp. 257–262, 1999.

[9] M.A. Abdou, A.A. Soliman, Variational iteration method for solving Burger's and coupled Burger'sequations*,* J. Comput. Appl. Math*.* 181 (2) pp. 245–251, 2005.

[10] G. Szego, Orthogonal polynomials, American Mathematical Society, New York, 1959 (revised edition).

[11] D. Gomez-Ullate, N. Kamran and R. Milson, ArXiv:nlin.SI/0401030

[12] W. Evans, Phys.Rev. A 41 pp. 5666- 5676, 1990.

[13] Shi-Hai Dong, G. Sun and M. Lozada-Cassou, Phys. Lett. A 340**,** pp. 94–103, 2005.

[14] Shi-Hai Dong, Physica Scripta, 65, pp. 289 -295, 2002.

[15] A.K. Chattopadhyay, Phys. Lett. A 357**,** pp**.** 108-111, 2006.

[16] C. Berkdemir, A. Berkdemir, J. Han, Chem. Phys. Lett. 417**,** pp. 326–329, 2006.

[17] Egrifes, H., Sever, R.: Phys. Lett. A 344, pp. 117 –126, 2005.

[18] Yasuk, F., Durmu¸s, A., Boztosun, I.: J. Math. Phys. 47, pp. 082302-9, 2006.

[19] J.-Y. Guo and Z.-Q. Sheng, Phys. Lett. A 338, pp. 90 –96, 2005.

[20] C. S. Jia, P. Gao, Y. F. Diao, L. Z. Yi, and X. J. Xie, Eur. Phys. J. A 34, pp. 41-48, 2007.

[21] Soylu, O. Bayrak, and I. Boztosun, J. Math. Phys. 48, pp. 082302-082309, 2007.

[22] F. Taşkın, Int. J. Theor. Phys. 48, pp. 1142 -1149, 2009.

[23] A. Soylu, O. Bayrak, and I. Boztosun, J. Phys. A: Math. Theor. 41, pp. 065308-15, 2008.

[24] C. S. Jia, P. Gao, and X. L. Peng, J. Phys. A 39, pp. 7737-7744, 2006.

[25] L. H. Zhang, X. P. Li, and C. S. Jia, Phys. Lett. A 372, pp. 2201- 2207. 2008.

[26] A. F. Nikiforov and V. B. Uvarov. Special Function of Mathematical Physics, Birkhauser, Bassel, 1988.

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