Abstract—A “clean” black hole is a black hole in vacuum such as the Schwarzschild black hole. However in real physical systems, there are matter fields around a black hole. Such a black hole is called a “dirty black hole”. In this paper, the effect of matter fields on the black hole and the greybody factor is investigated. The results show that matter fields make a black hole smaller. They can increase the potential energy to a black hole to obstruct Hawking radiation to propagate. This causes the greybody factor of a dirty black hole to be less than that of a clean black hole.

Keywords—A dirty black hole, Greybody factor, Hawking radiation, Matter fields.

I. INTRODUCTION

In general relativity, the spherically symmetric vacuum solution is the Schwarzschild metric. A Schwarzschild black hole, which is described by the Schwarzschild metric, is a black hole in empty space. It is referred to a “clean” black hole. However, in real situations a black hole is surrounded by various types of matter fields and referred to a “dirty” black hole [1]-[3]. We are interested in the presence of matter fields around a black hole. In this paper, the effect of matter fields on the black hole and the greybody factor is studied.

II. THE SCHWARZSCHILD BLACK HOLE

The Schwarzschild metric is given by

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2Mr/r} + r^2 d\Omega^2, \tag{1} \]

where \( M \) is the Schwarzschild black hole mass. In this paper, we are interested in a spin one particle radiated from the Schwarzschild black hole. We define the tortoise coordinate \( r_{\ast} \) by

\[ dr_{\ast} = \frac{dr}{1 - 2Mr/r}. \tag{2} \]

The Schwarzschild metric can be rewritten as

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)(-dt^2 + dr_{\ast}^2) + r_{\ast}^2 d\Omega^2. \tag{3} \]

The Schrödinger-like equation is given by

\[ \frac{d^2 \psi}{dr_{\ast}^2} + \left[\omega^2 - V(r)\right] \psi = 0, \tag{4} \]

where the potential for a vector field is [4]

\[ V(r) = \left(1 - \frac{2M}{r}\right)^{\frac{l(l+1)}{2}}. \tag{5} \]

The potential is plotted with \( l = 1 \) and \( M = 3 \) as shown in Fig. 1. In one-dimensional scattering problem, there are very general and rigorous bounds on the transmission probability [5]. Application for the generic systems can be found in [6]-[8]. For developments in applying to black hole greybody see [9], [10]. The lower bounds on the transmission probabilities are given by [5] (see also [11], [12])

\[ T \geq \text{sech}^2 \left(\int_{\infty}^{r_{\ast}} \vartheta dr_{\ast}\right), \tag{6} \]

where

\[ \vartheta = \frac{\sqrt{(h')^2 + (\omega^2 - V)^2}}{2h}, \tag{7} \]

for any positive function \( h \). We set \( h = \omega \), then

\[ T \geq \text{sech}^2 \left(\frac{1}{2\omega} \int_{-\infty}^{r_{\ast}} V dr_{\ast}\right). \tag{8} \]

From (2), the tortoise coordinate can be written as

\[ r_{\ast} = r + 2M \ln \left(\frac{r-2Mr}{2m}\right), \tag{9} \]

when \( r_{\ast} \to -\infty, r \to 2M \) and when \( r_{\ast} \to \infty, r \to \infty \). Therefore,

\[ T_{\text{sch}} \geq \text{sech}^2 \left(\frac{1}{2\omega} \int_{2M}^{r_{\ast}} \frac{l(l+1)}{r^2} dr\right) = \text{sech}^2 \left(\frac{l(l+1)}{4\omega M}\right). \tag{10} \]

The transmission probability and the energy of the emitted particle for the Schwarzschild black hole are plotted as shown in Fig. 2.
III. THE DIRTY BLACK HOLE

The metric of a generic static spherically symmetric spacetime is given by [1]

\[ ds^2 = -e^{-2\phi(r)} \left( 1 - \frac{2m(r)}{r} \right) dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2 d\Omega^2 \] (11)

where \( m(r) \) and \( \phi(r) \) are arbitrary functions. If the black hole event horizon exists, the function \( m(r) \) will satisfy the boundary condition \( 2m(r_H) = r_H \), where \( r_H \) is the horizon radius. Asymptotic flatness of spacetime is assumed so that \( \phi(\infty) = 0 \) and \( m(\infty) \) is finite. The Einstein equations are

\[ \frac{dm}{dr} = 4\pi \rho r^2 \quad \text{and} \quad \frac{d\phi}{dr} = -\frac{4\pi(\rho + p_r)r}{1-2m(r)/r} \] (12)

In this paper, we find an appropriate choice for \( m(r) \) and \( \phi(r) \) in order to derive the specific results. The Weak Energy Condition (WEC) states that \( \rho \geq 0 \). Thus the function \( m(r) \) is an increasing function. Because of finiteness of \( m(r) \) at spatial infinity, this motivate us to assume \( \rho r^4 = c_1 \), where \( c_1 \) is a constant. Therefore,

\[ m(r) = -\frac{4\pi c_1}{r} + c_2 \] (13)

where \( c_2 \) is a constant of integration. Applying the boundary condition \( 2m(r_H) = r_H \), we obtain

\[ c_2 = \frac{r_H}{2} + \frac{4\pi c_1}{r_H} \] (14)

Therefore,

\[ m(r) = \frac{r_H}{2} + 4\pi c_1 \left( \frac{1}{r_H} - \frac{1}{r} \right) \] (15)

Applying the boundary condition \( m(\infty) = M \), we obtain

\[ 4\pi c_1 = \frac{(2M-r_H)r_H}{2} \] (16)

Thus

\[ m(r) = M - \frac{(2M-r_H)r_H}{2} \] (17)

That is

\[ 1 - \frac{2m(r)}{r} = 1 - \frac{2M}{r} - \frac{(2M-r_H)r_H}{r^2} \] (18)

Since \( m(r) \) is an increasing function, then

\[ m(r_H) \leq m(\infty) \] (19)

or

\[ r_H \leq 2M. \] (20)

It can be seen that the horizon radius of the dirty black hole is smaller than that of the Schwarzschild black hole. Thus, matter fields around the black hole can shorten its horizon radius. The dirty black hole is, therefore, smaller than the Schwarzschild black hole for the same mass. From (12), we obtain

\[ \frac{d\phi}{dr} = -\frac{4\pi(\rho + p_r)r}{1-2m(r)/r^2} \] (21)

so

\[ \phi(r) = \ln \left( \frac{1 + x}{1 + x/\sqrt{y}} \right) + c_3, \] (22)

where

\[ x \equiv r - M - \sqrt{M^2 + A} \]

\[ y \equiv r - M + \sqrt{M^2 + A} \]

\[ a_2 \equiv \frac{A}{\left(\sqrt{M^2 + A} + M\right)\left(2\sqrt{M^2 + A}\right)} \]

\[ a_3 \equiv \frac{A}{\left(\sqrt{M^2 + A} - M\right)\left(2\sqrt{M^2 + A}\right)} \]

\[ A \equiv (2M - r_H)r_H \]
and $c_2$ is a constant of integration. Derivation of (22) can be seen in Appendix A. Applying the boundary condition $\phi(\infty) = 0$, we obtain $c_2 = 0$. Therefore, 

$$\phi(r) = \ln\left(\frac{r^{3/2}}{A^{3/2}L^{3/2}}\right).$$ (23)

The Schrödinger-like equation is given by

$$\frac{d^2\psi}{dr'^2} + \left[\omega^2 - V(r)\right]\psi = 0,$$ (24)

where

$$dr' = e^{\Phi(r)} \frac{dr}{1 - 2m(r)/r}.$$ (25)

and the potential for a vector field is [13]

$$V(r) = e^{-2\phi(r)} \left(1 - \frac{2m(r)}{r}\right) \frac{l(l+1)}{r^2}. \quad (26)$$

The dirty potential is plotted with $l = 1$, $M = 3$, and $r_H = 1$ as shown in Fig. 3. The comparison between the dirty potential and the Schwarzschild potential is plotted in Fig. 4. It can be seen that the dirty potential is higher than the Schwarzschild potential. Increment of the potential energy in case of the dirty black hole comes from matter fields. The lower bounds on the transmission probabilities are given by [13]

$$T_d \geq \text{sech}^2\left(\frac{1}{2} \int_{r_H}^\infty Vdr\right) = \text{sech}^2\left(\frac{1}{2} \int_{r_H}^\infty e^{-\phi(r)} \frac{l(l+1)}{r^2} dr\right) \geq \text{sech}^2\left(\frac{\Omega_{l+1}}{2\omega_{l+1}}\right). \quad (27)$$

Because of (20), we derive

$$T_d \leq T_{\text{sch}}.$$ (28)

Fig. 5 shows the plot between the transmission probability and the energy of the emitted particle for the dirty black hole with $l = 1$, $M = 3$, and $r_H = 1$. Fig. 6 shows the comparison between the transmission probabilities of the Schwarzschild black hole and the dirty black hole. The graph confirms the validity of (28).
Using (33), we can integrate (31)

\[ a_1 = -1 \]

\[ a_2 = \frac{A}{(\sqrt{M^2 + A} + A)(2\sqrt{M^2 + A})} \]

\[ a_3 = \frac{A}{(\sqrt{M^2 + A} - A)(2\sqrt{M^2 + A})} \]

equating the coefficients of powers of \( r \) gives

\[ a_1 = -1 \]

\[ a_2 = \frac{A}{(\sqrt{M^2 + A} + A)(2\sqrt{M^2 + A})} \]

\[ a_3 = \frac{A}{(\sqrt{M^2 + A} - A)(2\sqrt{M^2 + A})} \]

We define \( x \equiv r - M - \sqrt{M^2 + A} \) and \( y \equiv r + M + \sqrt{M^2 + A} \). Using (33), we can integrate (31)

\[ \phi(r) = -\frac{1}{r} \left( \frac{A r^2}{2} + \frac{a_2}{2} - \frac{a_3}{2} \right) \]

ACKNOWLEDGMENT

This research has been supported by a grant for the professional development of new academic staff from the Ratchadapisek Somphot Fund at Chulalongkorn University, by Thailand Toray Science Foundation (TTSF), by the Thailand Research Fund (TRF), the Office of the Higher Education Commission (OHEC), Chulalongkorn University, and by the Research Strategic Program (AI1B1), Faculty of Science, Chulalongkorn University. PB was additionally supported by a scholarship from the Royal Government of Thailand. TN was also additionally supported by a scholarship from the Development and Promotion of Science and Technology Talent Project (DPST). TN gives special thanks to Dr. Auttakit Chatrabuti for his invaluable advice. MV was additionally supported by the Marsden Fund, and by a James Cook fellowship, both administered by the Royal Society of New Zealand.

REFERENCES


