Construction Methods for Sign Patterns Allowing Nilpotence of Index $k$

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Abstract—In this paper, the smallest such integer $k$ is called by the index (of nilpotence) of $B$ such that $B^k = 0$. In this paper, we study sign patterns allowing nilpotence of index $k$ and obtain four methods to construct sign patterns allowing nilpotence of index at most $k$, which generalizes some recent results.

Keywords—Sign pattern, Nilpotence, Jordan block.

I. INTRODUCTION

The sign of a real number $a$, denoted by $\text{sgn}(a)$, is defined to be $1$, $-1$ or $0$, according to $a > 0$, $a < 0$, $a = 0$, respectively. A sign pattern matrix (or a sign pattern, for short) is a matrix whose entries are from the set $\{-1, 1, 0\}$. The sign pattern of a real matrix $B$, denoted by $\text{sgn}(B)$, is the sign pattern matrix obtained from $B$ by replacing each entry by its sign.

Let $m$ be a positive integer. The integers $a$ and $b$ are congruent modulo $m$ if and only if there is an integer $t$ such that $a = b + tm$ (for short, written as $a \equiv b \pmod{m}$).

Let $Q_n$ be the set of all sign patterns of order $n$. For $A \in Q_n$, the set of all real matrices with the same sign pattern as $A$ is called the qualitative class of $A$, and is denoted by $Q(A)$ ([2]).

Suppose that a real matrix has the property $p$. Then a sign pattern $A$ is said to require $p$ if every real matrix in $Q(A)$ has property $p$, or to allow $p$ if some real matrix in $Q(A)$ has property $p$ ([1]).

In this paper, we investigate the property $N$ of being nilpotent. Recall that a real matrix is said to be nilpotent if $B^k = 0$ for some positive integer $k$. The smallest such integer $k$ is called the index (of nilpotence) of $B$.

Let $k$ be a positive integer. We now consider sign patterns that allow nilpotence of index at most $k$. These sign patterns that allow nilpotence, are also referred to as the potentially nilpotent sign patterns (see [1], [4], [5], [6]). For convenience, we denote the class of all sign patterns that allow nilpotence of index at most $k$ by $N_k$. In [7], it is reported that it is an open problem to determine necessary and/or sufficient conditions for a sign pattern to allow nilpotence of index $k \geq 4$. Eschenbach and Li [4] studied $N_2$ and Gao, Li and Shao [1] studied $N_3$. In this paper, we mainly extend these results to any $N_k$.

II. PRELIMINARY

Lemma 1([4]). The set $N_k$ is closed under the following operations:

1) negation;
2) transposition;
3) permutational similarity, and
4) signature similarity.

As defined in [1], two sign patterns are equivalent if one can be obtained from the other by performing a sequence of operations listed in Lemma 1. This is indeed an equivalence relation.

Lemma 2 ([1]). A real matrix $B$ is nilpotent if and only if its eigenvalues are equal to zero.

Recall that a reducible (real or sign pattern) matrix is permutationally similar to a matrix in Frobenius normal form (see page 57 in [8]). Consequently, a reducible sign pattern $A$ allows nilpotence if and only if each irreducible component (see [8]) of $A$ allows nilpotence.

Lemma 3. Let $B$ be a nilpotent real matrix of index at most $k$, and $J$ the Jordan form of $B$. Then each Jordan block in $J$ is one of the following:

$$J_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \text{for } i = 2, 3, \ldots, k.$$

Let $A$ be a sign pattern matrix. The minimal rank of $A$, denoted by $\text{mr}(A)$, is defined as $\text{mr}(A) = \min\{\text{rank}B : B \in Q(A)\}$ ([3]).

Theorem 1. Let $A \in Q_n$. If $A \in N_k$, then

$$\text{mr}(A) \leq \frac{k - 1}{k - n}.$$  

Proof. Let $A \in Q_n$ and $A \in N_k$. Then there exists a real matrix $B \in Q(A)$ such that $B^k = 0$. By Lemma 3 we can assume that the Jordan form $J$ of $B$ is a direct sum of $k_i$ copies of $J_i$ ($i = 1, 2, \cdots, k$), where $\sum_{i=1}^k i k_i = n$. Then

$$\text{rank}(B) = \text{rank}(J) = \sum_{i=1}^k (i - 1)k_i \leq \frac{k-1}{k}k_1 + \frac{k-1}{k}2k_2 + \cdots + \frac{k-1}{k}kk_k = \frac{k-1}{k}n.$$  

Hence $\text{mr}(A) \leq \text{rank}(B) \leq \frac{k-1}{k}n$. \[ \square \]
Remark 1. Note that the sign pattern

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

satisfies $mr(A) = 3 \leq \frac{2}{3} \times 5$. However, $A^T \neq 0$, $A \notin N_4$. So the condition in Theorem 1 is not a sufficient one.

Theorem 2. Let $B$ be a real matrix of order $n$ with rank($B$) = $r$. Then $B^4 = 0$ if and only if there exist nonnegative integers $l$, $m$ and nonzero real column vectors $\alpha_1, \alpha_2, \cdots, \alpha_r$ and $\beta_1, \beta_2, \cdots, \beta_r$ of order $n$ with $l \leq \frac{n}{2}$, $m \leq \frac{n}{2}$, $2r - 2l - m \leq n$ and

$$\beta_j^T \alpha_i = \begin{cases} 1 & j \equiv 1 \pmod{3}, 1 \leq j \leq 3l - 1, \text{and } i = j + 1, \\
1 & j \equiv 2 \pmod{3}, 1 \leq j \leq 3l - 1, \text{and } i = j + 1, \\
otherwise, \end{cases}$$

such that

$$B = \sum_{1 \leq i \leq r} \alpha_i \beta_i^T.$$  \hfill (1)

Proof. Sufficiency. Let $B = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T$. By (1), we have

$$B^2 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)^2 = \alpha_1^2 \beta_1^2 + \alpha_2^2 \beta_2^2 + \cdots + \alpha_r^2 \beta_r^2,$$

$$B^3 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)^3 = \alpha_1^3 \beta_1^3 + \alpha_2^3 \beta_2^3 + \cdots + \alpha_r^3 \beta_r^3,$$

and

$$B^4 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)^4 = 0.$$

Necessity. Let $B^4 = 0$ with rank($B$) = $r$. By Lemma 3, the Jordan form $J$ of $B$ is a direct sum of $l$ copies of $J_l$, $m$ copies of $J_3$, $r - 3l - 2m$ copies of $J_2$ and $n - 4l - 3m - 2(r - 3l - 2m) = n - 2r + 2l + m$ copies of $J_1$, where $0 \leq l \leq \frac{n}{2}$, $0 \leq m \leq \frac{n}{2}$ and $2r - 2l - m \leq n$. It implies that there exists a nonsingular real matrix $D$ of order $n$ such that

$$D^{-1}BD = J = \begin{bmatrix} J_{11} & & \\ & J_{22} & \\ & & \ddots \\ & & & J_{n-r+2m+4l,n-r+2m+4l} \end{bmatrix},$$  \hfill (3)

where $J_{11} = \cdots = J_l = J_1$, $J_{1+l,1+l} = \cdots = J_{1+m,1+m} = J_3$, $J_{1+m+1,l+m+1} = \cdots = J_{r-m-2l+1,r-m-2l+1} = J_2$, $J_{r-m-2l+2,r-m-2l+2} = \cdots = J_{n-r+2m+4l,n-r+2m+4l} = J_1$.

Write

$$D = (u_1, u_2, \cdots, u_n) \text{ and } D^{-1} = (v_1, v_2, \cdots, v_n)^T,$$

where $u_1, u_2, \cdots, u_n$ are column vectors of $D$ and $v_1, v_2, \cdots, v_n$ are column vectors of $D^{-1}$. Clearly, $v_i^T u_i = 1$, for $i = 1, 2, \cdots, n$, and $v_i^T u_j = 0$, for $i \neq j$. Let $\alpha_{3i-2} = u_{4i-3}, \alpha_{3i-1} = u_{4i-2}, \alpha_{3i} = u_{4i+1}$ for $i = 1, 2, \cdots, l$, $\alpha_{3l+2j-1} = u_{4i+3j-2}, \alpha_{3l+2j} = u_{4i+3j+1}$, for $j = 1, 2, \cdots, m$, $\alpha_{3l+2m+s} = u_{4l+3m+2s-1}$, for $s = 1, 2, \cdots, r - 3l - 2m$, $\beta_{3l+2j-1} = v_{4i+3j-2}, \beta_{3l+2j} = v_{4l+3j+1}$, for $j = 1, 2, \cdots, l$, $\beta_{3l+2m+s} = v_{4l+3m+2s-1}$, for $s = 1, 2, \cdots, r - 3l - 2m$.

It is easy to see that $\alpha_i$ and $\beta_i$ satisfy the condition (1). By (3), we have

$$B = DJD^{-1} = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T.$$  \hfill \hfill (4)

The conclusion follows. $\square$

Next, we generalize the above result to any $B^k = 0$, that is, $N_k$.

Lemma 3. Let $B$ be a real matrix of order $n$ with rank($B$) = $r$. Then $B^k = 0$ if and only if there exist nonnegative integers $l_1, l_2, \cdots, l_k$ and nonzero real column vectors $\alpha_1, \alpha_2, \cdots, \alpha_r$ and $\beta_1, \beta_2, \cdots, \beta_r$ of order $n$ with $\sum_{i=1}^k l_i = n$, $\sum_{i=1}^k (i - 1)l_i = r$, and

$$\beta_j^T \alpha_i = \begin{cases} 1, & j \equiv s(\pmod{k-1}), s = 1, 2, \cdots, k-2, \\
1 & l \leq j \leq (k-1)l_i + 1, \text{and } i = j + 1, \\
otherwise, \end{cases}$$

such that

$$B = \sum_{1 \leq i \leq r} \alpha_i \beta_i^T.$$  \hfill (5)

Proof. Sufficiency. Let $B = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T$. By (4), we have

$$B^2 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)^2 = (\alpha_1 \beta_1^T)^2 + \cdots + (\alpha_r \beta_r^T)^2 = (\alpha_1 \beta_1^T)^2 + \cdots + (\alpha_r \beta_r^T)^2,$$

$$B^3 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)^3 = (\alpha_1 \beta_1^T)^3 + \cdots + (\alpha_r \beta_r^T)^3,$$

and

$$B^4 = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)^4 = 0.$$
and

\[ B^k = B B^{k-1} = (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T)(\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_k \beta_k^T) + \cdots + \alpha_{(k-2)} \beta_{(k-1)}^T (\beta_l)^T \]

\[ = 0. \]

Necessity. Let \( B^k = 0 \) with rank(\( B \)) = \( r \). By Lemma 3, the Jordan form of \( B \) is a direct sum of \( l_i \) copies of \( J_i \), where \( \sum_{i=1}^k l_i = n \), \( \sum_{i=1}^k (i-1)l_i = r \) and it implies that there exists a nonsingular real matrix \( D \) of order \( n \) such that

\[
D^{-1} B D = J = \begin{bmatrix}
J_{11} & & \\
& J_{22} & \\
& & \ddots \\
& & & J_{n-r,n-r}
\end{bmatrix},
\]

where

\[ J_{11} = \cdots = J_{l_k l_k} = J_k, \]

\[ J_{l_k+1,l_k+1} = \cdots = J_{l_k+l_k-1,l_k+l_k-1} = J_{k-1}, \]

\[ \cdots, \]

\[ J_{\sum_{i=2}^k l_i+1,\sum_{i=2}^k l_i} = \cdots = J_{n-r,n-r} = J_1. \]

Write

\[ D = (u_1, u_2, \ldots, u_n) \text{ and } D^{-1} = (v_1, v_2, \ldots, v_n)^T, \]

where \( u_1, u_2, \ldots, u_n \) are column vectors of \( D \) and \( v_1, v_2, \ldots, v_n \) are column vectors of \( D^{-1} \), \( v_i^T u_i = 1 \), for \( i = 1, 2, \ldots, n \), and \( v_i^T u_i = 0 \), for \( i \neq j \). Let

\[ \alpha_i = \alpha_{(s-1)i+m} + \sum_{j=s+1}^k \beta_{ij}, \]

\[ \beta_i = \alpha_{(s-1)i+m} + \sum_{j=s+1}^k \beta_{ij}, \]

for \( s = 1, 2, \ldots, k \), \( i = 1, 2, \ldots, l_s - 1 \), and \( m = 1, 2, \ldots, s - 1 \). It is easy to see that \( \alpha_i \) and \( \beta_i \) satisfy the condition (4). By (6), we have that

\[ B = D J D^{-1} = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \cdots + \alpha_r \beta_r^T. \]

The proof is completed. \( \Box \)

III. MAIN RESULTS

Based on the above analysis, one can obtain the following construction methods to find a sign pattern in \( N_k \).

A. Construction Method 1—Jordan Method

By Lemma 3, we may obtain the Jordan form method to construct a sign patterns in \( N_k \). For example, let

\[
J = \begin{bmatrix}
J_1 & 0 \\
0 & J_2
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
1 & 2 & 2 & 0 & 1 & 3 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 2 \\
1 & 0 & 2 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
J_1 = \begin{bmatrix}
-1 & -1 & -2 & 2 & 2 & 3 \\
0 & 1 & 0 & 0 & -1 & 1 \\
1 & 1 & 1 & -1 & -2 & -2 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
D^{-1} = \begin{bmatrix}
0 & 2 & 2 & 0 & -2 & -1 \\
1 & 0 & 1 & -1 & -1 & -1 \\
0 & 1 & 1 & 0 & -1 & -1 \\
-2 & 1 & 0 & 2 & 1 & 3 \\
1 & 0 & 1 & -1 & -1 & -2 \\
1 & 1 & 1 & -1 & -1 & -2
\end{bmatrix},
\]

\[
B = DJD^{-1} = \begin{bmatrix}
0 & 1 & 1 & 0 & -1 & -1 \\
0 & 1 & 1 & 0 & -1 & -1 \\
-1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1
\end{bmatrix},
\]

Note that

\[
A = \text{sgn}(B) = \begin{bmatrix}
1 & 1 & 1 & 0 & -1 & -1 \\
1 & 0 & 1 & -1 & -1 & -1 \\
0 & 1 & 1 & 0 & -1 & -1 \\
-1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1
\end{bmatrix}.
\]

B. Construction Method 2—vectors spanning method

Let \( l_1, l_2, \ldots, l_k \) be nonnegative integers with \( \sum_{i=1}^k l_i = n \), \( \sum_{i=1}^k (i-1)l_i = r \). Let real column vectors \( \alpha_1, \alpha_2, \ldots, \alpha_r \) and \( \beta_1, \beta_2, \ldots, \beta_r \) of order \( n \) satisfy the condition (7).

\[
\beta_i = \begin{cases}
1 & j = s (\text{mod } k-1), s = 1, 2, \ldots, k-2, \\
1 & j = (k-1)l_k - 1, i = j + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

By Theorem 3, the real matrix

\[
B = \sum_{i=1}^{l_k} \alpha_i \beta_i^T
\]

is nilpotent of index at most \( k \), and its sign pattern is in \( N_k \). For example, let \( n = 8 \), \( r = 6 \), \( l = m = 1 \),

\[
\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}.
\]
\[ \alpha_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \alpha_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad \alpha_6 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}, \]

\[ \beta_1 = (1, 1, 0, -1, 0, 0, 1, -1), \quad \beta_2 = (-1, 1, 2, -1, 1, -1, 0, 1), \quad \beta_3 = (-1, 0, 0, 0, 0, 0, 0, 1), \quad \beta_4 = (0, 0, -1, 1, 0, -1, 0, 0), \quad \beta_5 = (1, -1, -1, 1, -1, 0, 0, 0), \quad \beta_6 = (-1, 0, 0, 0, 0, 0, 0, 1). \]

\[ B = \sum_{1 \leq i \leq 6} \alpha_i \beta_i = \begin{bmatrix} 1 & 2 & 1 & -3 & 1 & 0 & -4 & 1 \\ 2 & 5 & 5 & -4 & 4 & 2 & -8 & 1 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & 3 & 2 & -1 & -3 & 1 \\ 3 & 1 & 1 & -3 & 1 & 0 & -4 & 0 \\ -1 & 2 & 3 & -2 & 3 & 2 & -4 & 3 \\ 1 & 2 & 2 & -3 & 1 & 0 & -4 & 1 \\ 1 & 2 & 2 & -3 & 1 & 0 & -4 & 1 \end{bmatrix} \]

\[ D^4 = 0. \]

Then

\[ A = \text{sgn} (B) \in N_4. \]

**C. Construction Method 3—block method**

**Theorem 4.** Suppose \( A = [A_1 \ A_2 \ A_3 \ A_4] \in N_k \), where \( A_1 \) and \( A_4 \) are square, then for any positive integer \( m \), we have

\[ \tilde{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_2 \\ A_3 & A_4 & \cdots & A_4 \\ \vdots & \vdots & \ddots & \vdots \\ A_3 & A_4 & \cdots & A_4 \end{bmatrix} \in N_k, \]

where \( \tilde{A} \) has \((m+1)^2\) blocks.

**Proof.** Note that, if \( A \in N_k \), then there is a real matrix

\[ B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \in Q(A), \]

where \( B_i \in Q(A_i) \), such that \( B^{k} = 0 \).

Let

\[ B^j = \begin{bmatrix} f_{1j}(B_1, B_2, B_3, B_4) & f_{2j}(B_1, B_2, B_3, B_4) \\ f_{3j}(B_1, B_2, B_3, B_4) & f_{4j}(B_1, B_2, B_3, B_4) \end{bmatrix} \]

for \( j = 1, 2, \cdots, k - 1 \).

For short, we denote

\[ B^j = \begin{bmatrix} f_{1j} & f_{2j} \\ f_{3j} & f_{4j} \end{bmatrix} \]

for \( j = 1, 2, \cdots, k - 1 \). And

\[ \tilde{B} = \begin{bmatrix} B_1 & \frac{1}{m} B_2 & \cdots & \frac{1}{m} B_2 \\ B_3 & \frac{1}{m} B_4 & \cdots & \frac{1}{m} B_4 \\ \vdots & \vdots & \ddots & \vdots \\ B_3 & \frac{1}{m} B_4 & \cdots & \frac{1}{m} B_4 \end{bmatrix}. \]

When \( k = 2 \), it follows that

\[ B^2 = \begin{bmatrix} B_1^2 + B_2 B_3 & B_1 B_2 + B_2 B_4 \\ B_2 B_1 + B_3 B_4 & B_2 B_3 + B_4 B_2 \end{bmatrix}. \]

\[ \tilde{B}^2 = \begin{bmatrix} B_1^2 + B_2 B_3 & \frac{1}{m} (B_1 B_2 + B_2 B_4) & \cdots & \frac{1}{m} (B_1 B_2 + B_2 B_4) \\ \frac{1}{m} (B_1 B_2 + B_2 B_4) & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m} (B_1 B_2 + B_2 B_4) & \cdots & \cdots & \frac{1}{m} (B_1 B_2 + B_2 B_4) \end{bmatrix}. \]

So \( \tilde{B}^2 = 0 \). Thus \( \tilde{A} \in N_k \).

Suppose that we have

\[ \tilde{B}^s , \tilde{B}^s \begin{bmatrix} f_{1s} & \frac{1}{m} f_{2s} & \cdots & \frac{1}{m} f_{2s} \\ f_{3s} & \frac{1}{m} f_{4s} & \cdots & \frac{1}{m} f_{4s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{3s} & \frac{1}{m} f_{4s} & \cdots & \frac{1}{m} f_{4s} \end{bmatrix} \text{ for } 2 \leq s < k, \]

then

\[ B^{s+1} = B^s = \begin{bmatrix} f_{1s} & f_{2s} \\ f_{3s} & f_{4s} \\ \vdots & \vdots \\ f_{3s} & f_{4s} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} f_{1s} B_1 + f_{2s} B_3 & f_{1s} B_2 + f_{2s} B_4 \\ f_{3s} B_1 + f_{4s} B_3 & f_{3s} B_2 + f_{4s} B_4 \end{bmatrix} \]

and

\[ \tilde{B}^{s+1} = \begin{bmatrix} f_{1s} & \frac{1}{m} f_{2s} & \cdots & \frac{1}{m} f_{2s} \\ f_{3s} & \frac{1}{m} f_{4s} & \cdots & \frac{1}{m} f_{4s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{3s} & \frac{1}{m} f_{4s} & \cdots & \frac{1}{m} f_{4s} \end{bmatrix} \begin{bmatrix} B_1 & \frac{1}{m} B_2 & \cdots & \frac{1}{m} B_2 \\ B_3 & \frac{1}{m} B_4 & \cdots & \frac{1}{m} B_4 \end{bmatrix} = \begin{bmatrix} f_{1s} B_1 + f_{2s} B_3 & \frac{1}{m} (f_{1s} B_2 + f_{2s} B_4) & \cdots & \frac{1}{m} (f_{1s} B_2 + f_{2s} B_4) \\ f_{3s} B_1 + f_{4s} B_3 & \frac{1}{m} (f_{3s} B_2 + f_{4s} B_4) & \cdots & \frac{1}{m} (f_{3s} B_2 + f_{4s} B_4) \\ \vdots & \vdots & \ddots & \vdots \\ f_{3s} B_1 + f_{4s} B_3 & \frac{1}{m} (f_{3s} B_2 + f_{4s} B_4) & \cdots & \frac{1}{m} (f_{3s} B_2 + f_{4s} B_4) \end{bmatrix} \]

So

\[ B^k = \begin{bmatrix} f_1(k-1) B_1 + f_2(k-1) B_3 & f_1(k-1) B_2 + f_2(k-1) B_4 \\ f_3(k-1) B_1 + f_4(k-1) B_3 & f_3(k-1) B_2 + f_4(k-1) B_4 \end{bmatrix} \in N_k. \]

\[ \tilde{B}^k = 0. \]

By the principle of mathematical induction, we have \( \tilde{A} \in N_k. \)

**D. Construction Method 4—null space method**

**Theorem 5.** Let \( B \) and \( C \) be nilpotent real matrices of indices of \( k \) with order \( v_1 \) and \( v_2 \), respectively. Let \( p \) be a positive integer. The kernel of a matrix \( B \), denoted by \( \text{Ker}(B) \), also called the null space, is the kernel of the linear map defined by the matrix \( B \). Suppose that the following conditions hold:

\[ u_1, u_2, \cdots, u_p \in \text{Ker}(B^j), \quad v_1, v_2, \cdots, v_p \in \text{Ker}((C^j)^T), \]

(9)
where $1 \leq i < k$, $1 \leq j < k$, and $i + j \leq k$. Then the following partitioned block real matrix of order $n_1 + n_2$

$$D = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$$

is nilpotent of index at most $k$ and $A = \text{sgn}(D) \in \mathbb{N}_k$, where $X = u_1v_1^T + u_2v_2^T + \cdots + u_pv_p^T$.

**Proof.** In fact, let $D = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$, where $B$ and $C$ are square. Then

$$D^k = \begin{bmatrix} B^k & B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} \\ 0 & C^k \end{bmatrix}.$$ 

Thus $D^k = 0$ if and only if $B^k = 0$, $C^k = 0$ and $B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} = 0$.

It is obvious that $B^k = 0$ and $C^k = 0$. In addition, we observe that

$$B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} = B^{k-2}(Bu_1v_1^T + Bu_2v_2^T + \cdots + Bu_pv_p^T)$$

$$+ B^{k-3}(u_1, \ldots, u_p) \begin{pmatrix} v_1^T \\ \vdots \\ v_p^T \end{pmatrix} C$$

$$+ \cdots + (u_1, \ldots, u_p) \begin{pmatrix} v_1^T \\ \vdots \\ v_p^T \end{pmatrix} C^{k-1}.$$ 

Therefore, we get the desired result with the above condition (9). $\square$

**IV. Conclusion**

In this paper, sign patterns allowing nilpotence of index at most $k$ are researched and four methods to construct sign patterns under the condition that allows nilpotence of index at most $k$ are obtained, which generalizes some recent results in [1], [4] and has a certain theoretical and practical value.

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