The Existence and Uniqueness of Positive Solution for Nonlinear Fractional Differential Equation Boundary Value Problem

Chuanyun Gu, Shouming Zhong

Abstract—In this paper, the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problem is concerned by a fixed point theorem of a sum operator. Our results can not only guarantee the existence and uniqueness of positive solution, but also be applied to construct an iterative scheme for approximating it. Finally, the example is given to illustrate the main result.

Keywords—Fractional differential equation, Boundary value problem, Positive solution, Existence and uniqueness, Fixed point theorem of a sum operator.

I. INTRODUCTION

Fractional differential equations are used in various fields, such as mechanics, physics, chemistry, engineering, economics and biological sciences, etc.; see [1 – 9] and the references therein. In recent years, there are many papers discuss the existence and multiplicity of positive solutions for nonlinear fractional differential equation boundary value problem by the use of Leray-Shauder theory, fixed-point theorems, etc., see [10 – 14]. However, there are few papers consider the existence of a unique positive solution for nonlinear fractional differential equation boundary value problem, see [15, 16].

In particular, by means of mixed monotone method, Xu, Jiang and Yuan [16] obtained the uniqueness of solution to the singular boundary value problem for fractional differential equation

\[
\left\{ \begin{array}{l}
D_0^+ u(t) = f(t, u(t)), \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \\
   u(0) = u(1) = u'(0) = u'(1) = 0.
\end{array} \right.
\]  

They need

\[
f(t, u) = q(t)[\varphi(u) + \psi(u)], \varphi : [0, +\infty) \to [0, +\infty)
\]

is continuous and nondecreasing, \( \psi : (0, +\infty) \to (0, +\infty) \) is continuous and nondecreasing and \( q \in C([0, 1], (0, +\infty)) \) satisfies

\[
\int_0^1 s^{\eta - 2(\alpha - 1)}(1 - s)^{\alpha - 2 - \eta}q(s)ds < +\infty, \quad \eta \in (0, 1).
\]

Naturally, when \( f(t, u) \) can not be denoted by \( f(t, u) = q(t)[\varphi(u) + \psi(u)] \), how to consider the unique positive solution of it? In this paper, by means of a fixed point theorem for a sum operator, we obtain the existence and uniqueness of positive solution to the nonsingular boundary value problem for fractional differential equation (1) with the assumption \( f(t, u) = g(t, u) + h(t, u) \). Moreover, we can construct an iterative scheme to approximate the unique solution, which is important for evaluation and application.

II. PRELIMINARIES AND PREVIOUS RESULTS

In this section, we present some definitions, lemmas and basic results that will be used in the proof of our main result.

Definition 1: The integral

\[
P_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{\alpha-1}}dt, \quad x > 0
\]

is called the Riemann-Liouville fractional integral of order \( \alpha \), where \( \alpha > 0 \) and \( \Gamma(\alpha) \) denotes the gamma function.

Definition 2: For a function \( f(x) \) given in the interval \([0, \infty)\), the expression

\[
D_0^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(t)}{(x-t)^{\alpha+n-1}}dt,
\]

is called the Riemann-Liouville fractional derivative of order \( \alpha \), where \( n = [\alpha] + 1, [\alpha] \) denotes the integer part of number \( \alpha \).

Lemma 1: Given \( y \in C([0, 1]) \) and \( 3 < \alpha \leq 4 \), the unique solution of the fractional differential equation

\[
\left\{ \begin{array}{l}
D_0^\alpha u(t) = y(t), \quad 0 < t < 1, \\
   u(0) = u(1) = u'(0) = u'(1) = 0,
\end{array} \right.
\]

is

\[
u(t) = \int_0^t G(t, s)y(s)ds, \quad t \in [0, 1]
\]

where

\[
G(t, s) = \left\{ \begin{array}{ll}
\frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \frac{(s-t)+(\alpha-2)(1-t)s}{1-t}, & 0 \leq s \leq t \leq 1, \\
\frac{e^{-2(s-t)}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \frac{(s-t)+(\alpha-2)(1-t)s}{1-t}, & 0 \leq t \leq s \leq 1.
\end{array} \right.
\]

Here \( G(t, s) \) is called the Green function of the fractional differential equation (2).
Lemma 2 [16] The Green function $G(t, s)$ defined by (3) has the following property:

$$G(t, s) = \frac{1}{\Gamma(\alpha)}(s \leq t) \frac{1}{(1-s)^{\alpha-2}}(1-t)^{\alpha-2} G(t, s) \leq \frac{1}{\Gamma(\alpha)}M_0 t^{\alpha-2}(1-t)^2 \tag{4}$$

for $t, s \in (0, 1)$, where $M_0 = \max\{\alpha-1, (\alpha-2)^2\}$.

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and a fixed point theorem which will be used later.

Suppose $(E, || \cdot ||)$ is a real Banach space which is partially ordered by a cone $P \subseteq E$, i.e. $x \leq y$ if and only if $y-x \in P$. If $x \leq y$ and $y \neq x$, then we denote $x < y$. We denote the zero element of $E$ by $\theta$. Recall that a non-empty closed convex set $P \subseteq E$ is a cone if it satisfies (i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$.

Putting $P^0 = \{x \in P | x$ is an interior point of $P\}$, a cone $P$ is said to be solid if $P^0$ is non-empty. Moreover, $P$ is called normal if there exists a constant $N > 0$ such that for all $x, y \in E, \theta \leq x \leq y$ implies $||x|| \leq N||y||$; in this case $N$ is called the normality constant of $P$. We say that an operator $A : E \to E$ is increasing if $x \leq y$ implies $Ax \leq Ay$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly $\sim$ is an equivalence relation. Given $w > \theta$ (i.e. $w \geq \theta$ and $w \notin \theta$), we denote the set $P_w = \{x \in E | x \sim w\}$ by $P_w$. It is easy to see that $P_0 \subseteq P_w$ for $w \in P$.

Theorem 1 [17] Let $P$ be a normal cone in a real Banach space $E$, $A : P \to P$ be an increasing $\gamma$-concave operator and $B : P \to P$ be an increasing sub-homogeneous operator. Assume that

(i) there is $w > \theta$ such that $Aw \in P_w$ and $Bw \in P_w$;

(ii) there exists a constant $\delta_0 > 0$ such that $Ax \geq \delta_0 Bx$ for all $x \in P$.

Then operator equation $Ax + Bx = x$ has a unique solution $x^*$ in $P_w$. Moreover, constructing successively the sequence $y_0 = Ay_{n-1} + By_{n-1}, n = 1, 2, \cdots$ for any initial value $y_0 \in P_w$, we have $y_n \to x^*$ as $n \to \infty$.

III. MAIN RESULTS

In this section, we apply Theorem 1 to investigate the fractional differential equation (1) with the assumption $f(t, u) = g(t, u) + h(t, u)$, and we obtain the new result on the existence and uniqueness of positive solution.

In this paper, we will work in the Banach space $C[0, 1] = \{x : [0, 1] \to R \text{ is continuous} \}$ with the standard norm $||x|| = \sup \{|x(t)| : t \in [0, 1]\}$. Notice that this space can be endowed with a partial order given by $x, y \in C[0, 1]$, $x \leq y$ if and only if $x(t) \leq y(t)$ for $t \in [0, 1]$.

Let $P = \{x \in C[0, 1] | x(t) \geq 0, t \in [0, 1]\}$ be the standard cone. Evidently, $P$ is a normal cone in $C[0, 1]$ and the normality constant is 1.

Theorem 2 Assume that

(H1) $g, h : [0, 1] \times [0, \infty) \to [0, \infty)$ are continuous and increasing with respect to the second argument, $h(t, 0) \neq 0$;

(H2) there exists a constant $\gamma \in (0, 1)$ such that $g(t, x) \geq \gamma g(t, x), \forall t \in [0, 1], x \in [0, 1]$, and $h(t, x) \geq \mu h(t, x)$ for $\mu \in (0, 1), t \in [0, 1], x \in [0, \infty)$;

(H3) there exists a constant $\delta_0 > 0$ such that $g(t, x) \geq \delta_0 h(t, x), t \in [0, 1], x \geq 0$. Then the fractional differential equation (1) has a unique positive solution $u^*$ in $P_w$, where $w(t) = t^{\alpha-2}(1-t)^2, t \in [0, 1]$. Moreover, for any initial value $u_0 \in P_w$, constructing successively the iterative scheme

$$u_{n+1}(t) = \int_0^1 G(t, s)f(s, u_n(s))ds, \quad n = 0, 1, 2, \cdots,$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G(t, s)$ is given as (3).

Proof: To begin with, from Lemma 1, the fractional differential equation (1) has an integral formulation by

$$u(t) = \int_0^1 G(t, s)f(s, u(s))ds = \int_0^1 G(t, s)g(s, u(s)) + h(s, u(s))ds$$

where $G(t, s)$ is given as in Lemma 1.

Define two operators $A : P \to E$ and $B : P \to E$ by

$$Au(t) = \int_0^1 G(t, s)g(s, u(s))ds, \quad Bu(t) = \int_0^1 G(t, s)h(s, u(s))ds.$$

It is easy to prove that $u$ is the solution of the fractional differential equation (1) if and only if $u = Au + Bu$. By assumption (H1) and Lemma 2, we know that $A : P \to P$ and $B : P \to P$. In the sequel we check that $A, B$ satisfy all assumptions of Theorem 1.

Firstly, we prove that $A$ is a two increasing operators.

In fact, from assumption (H1) and Lemma 2, for $u, v \in P$ with $u \geq v$, we know that $u(t) \geq v(t), t \in [0, 1]$ and obtain

$$Au(t) = \int_0^1 G(t, s)g(s, u(s))ds \geq \int_0^1 G(t, s)g(s, v(s))ds = Av(t).$$

That is $Au \geq Av$. Similarly, $Bu \geq Bv$.

Next we show that $A$ is a $\gamma$-concave operator and $B$ is a sub-homogeneous operator.

In fact, for any $\lambda \in (0, 1)$ and $u \in P$, from (H2) we know that

$$A(\lambda u)(t) = \int_0^1 G(t, x)g(s, \lambda u(s))ds \geq \lambda^{\gamma} \int_0^1 G(t, s)g(s, u(s))ds = \lambda^{\gamma} Au(t).$$

That is, $A(\lambda u) \geq \lambda^{\gamma} Au$ for $\lambda \in (0, 1), u \in P$. So the operator $A$ is a $\gamma$-concave operator. Also, for any $\mu \in (0, 1)$ and $\mu \in P$, by (H2) we obtain

$$B(\mu u)(t) = \int_0^1 G(t, x)h(s, \mu u(s))ds \geq \mu^{\gamma} \int_0^1 G(t, s)h(s, u(s))ds = \mu Bu(t).$$

That is, $B(\mu u) \geq \mu Bu$ for $\mu \in (0, 1), u \in P$. So the operator $B$ is a sub-homogeneous operator.

Now we show that $Aw \in P_w$ and $Bw \in P_w$, where $w(t) = t^{\alpha-2}(1-t)^2$.

By (H1) and Lemma 2,

$$\int_0^1 \frac{1}{\Gamma(\alpha)} \int_0^1 G(t, s)g(s, w(s))ds \leq \int_0^1 w(t)g(t, s)ds \leq \frac{1}{\Gamma(\alpha)} \int_0^1 M_0 g(s, 1)ds,$$

$$\int_0^1 \frac{1}{\Gamma(\alpha)} \int_0^1 G(t, s)h(s, w(s))ds \leq \int_0^1 w(t)h(t, s)ds \leq \frac{1}{\Gamma(\alpha)} \int_0^1 M_0 h(s, 1)ds.$$
We present one example to illustrate Theorem 2.

**Example 1** Consider the following fractional differential equation:

\[
\begin{cases}
P_{0+}^{\alpha} u(t) = u^2(t) + \arctan u(t) + t^3 + t^2 + \frac{\pi}{7}, \\
u(0) = u(0) = u'(0) = u''(0) = 0.
\end{cases}
\] (5)

In this example, we have \( \alpha = \frac{10}{3} \). Let

\[
g(t, u) = u^2(t) + t^3 + \frac{\pi}{7},
\]

\[
h(t, u) = \arctan u(t) + t^3,
\]

\[
\gamma = \frac{1}{5}.
\]

Obviously, \( g, h : [0, 1] \times [0, \infty) \to [0, \infty) \) are continuous and increasing with respect to the second argument, \( h(t, 0) = t^3 \neq 0 \).

Furthermore, for \( t \in [0, 1], \lambda \in (0, 1), x \in [0, \infty) \), we have

\[
g(t, \lambda u) = \lambda^2 u^2(t) + t^3 + \frac{\pi}{7}
\]

\[
\geq \lambda^2 u^2(t) + t^3 + \frac{\pi}{7} + \arctan u
\]

\[
= \lambda^2 g(t, u)
\]

and for \( t \in [0, 1], \mu \in (0, 1), x \in [0, \infty) \), we have

\[
\arctan(\lambda u) \geq \mu \arctan u,
\]

and thus

\[
h(t, \mu u) \geq \mu h(t, u).
\]

Moreover, if we take \( \delta_0 \in (0, 1) \), then we obtain

\[
g(t, u) = u^2(t) + t^3 + \frac{\pi}{7}
\]

\[
\geq t^3 + \frac{\pi}{7}
\]

\[
\geq t^3 + \arctan u
\]

\[
\geq \delta_0 (t^3 + \arctan u)
\]

\[
= \delta_0 h(t, u)
\]

Hence all the conditions of Theorem 2 are satisfied. An application of Theorem 2 implies that the fractional differential equation (5) has a unique positive solution in \( P_\omega \), where \( u(0) = u'(0) = u''(0) = 0 \).

**Remark 2** The nonlinearity \( f \) in example 1 can not be denoted by \( f(t, u) = g(t)[\varphi(u) + \psi(u)] \), so the positive solution of example 1 can not be obtained by virtue of [16].Theorem 2 generalizes the results in [16].

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**REFERENCES**


Chuanyun Gu was born in Henan Province, China, in 1982. He received the B.S. degree from Nanyang Normal University, Nanyang, in 2008; he received the M.S. degree in applied mathematics from Xihua University, Chengdu, in 2012. His research interests include differential equations and functional analysis.

Shouming Zhong was born in 1955 in Sichuan, China. He received B.S. degree in applied mathematics from UESTC, Chengdu, China, in 1982. From 1984 to 1986, he studied at the Department of Mathematics in Sun Yatsen University, Guangzhou, China. From 2005 to 2006, he was a visiting research associate with the Department of Mathematics in University of Waterloo, Waterlo, Canada. He is currently as a full professor with School of Applied Mathematics, UESTC. His current research interests include differential equations, neural networks, biomathematics and robust control. He has authored more than 80 papers in reputed journals such as the International Journal of Systems Science, Applied Mathematics and Computation, Chaos, Solitons and Fractals, Dynamics of Continuous, Discrete and Impulsive Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Acta Electronica Sinica, Control and Decision, and Journal of Engineering Mathematics.