

Intuitionistic Fuzzy Implicative Ideals with Thresholds (λ, μ) of BCI-Algebras

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Abstract—The aim of this paper is to introduce the notion of intuitionistic fuzzy implicative ideals with thresholds (λ, μ) of BCI-algebras and to investigate its properties and characterizations.

Keywords—BCI-algebra, intuitionistic fuzzy set, intuitionistic fuzzy ideal with thresholds (λ, μ) , intuitionistic fuzzy implicative ideal with thresholds (λ, μ) .

I. INTRODUCTION

A BCI-algebra is an important class of logical algebra and was introduced by Iséki [1], [2]. K. Atanassov [3] introduced the concept of intuitionistic fuzzy sets. At present this concept has been applied to many mathematical branches. In 2003, K. Hur [4] applied the concept to the theory of rings, and introduced the concepts of intuitionistic fuzzy subgroups and subrings. M. Jiang and X.L. Xin [5] later introduced the concepts of (λ, μ) intuitionistic fuzzy subrings (Ideals), some meaningful results are obtained. In [6], we have given the concepts of intuitionistic fuzzy subalgebras with thresholds (λ, μ) and intuitionistic fuzzy ideals with thresholds (λ, μ) of BCI-algebras, in this paper, we apply the concept of intuitionistic fuzzy sets to the ideals theory of BCI-algebras, and introduce the notions of intuitionistic fuzzy implicative ideals with thresholds (λ, μ) of BCI-algebras. We give several properties and characterizations of intuitionistic fuzzy implicative ideals with thresholds (λ, μ) of BCI-algebras.

II. PRELIMINARIES

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following axioms:

$$(BCI-1) \quad ((x * y) * (x * z)) * (z * y) = 0,$$

$$(BCI-2) \quad (x * (x * y)) * y = 0,$$

$$(BCI-3) \quad x * x = 0,$$

$$(BCI-4) \quad x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y,$$

for all $x, y, z \in X$. In a BCI-algebra X , we can define a partial ordering \leq by putting $x \leq y$ if and only if $x * y = 0$.

In any BCI-algebra X , the following hold:

- (1) $(x * y) * z = (x * z) * y$,
 - (2) $x * 0 = x$,
 - (3) $0 * (x * y) = (0 * x) * (0 * y)$,
 - (4) $(x * z) * (y * z) \leq x * y$,
 - (5) $x * (x * (x * y)) = x * y$,
 - (6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,
- for all $x, y, z \in X$.

In this paper, X always means a BCI-algebra unless otherwise specified.

A nonempty subset K of X is called an ideal of X if $(I_1) : 0 \in K, (I_2) : x * y \in K$ and $y \in K$ imply $x \in K$. A nonempty subset K of X is called a implicative ideal of X if it satisfies (I_1) and $(I_3) : (((x * y) * y) * (0 * y)) * z \in K$ and $z \in K$ imply $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in K$.

Definition 1 [3] Let S be any set. An intuitionistic fuzzy subset A of S is an object of the following form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \} \text{ where } \mu_A : S \rightarrow [0, 1]$$

and $\nu_A : S \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in S$ respectively and for every $x \in S, 0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

$$\text{Denote } \langle I \rangle = \{ \langle a, b \rangle : a, b \in [0, 1] \}.$$

Definition 2 Let $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \}$ be an intuitionistic fuzzy set in a set S . For $\langle \alpha, \beta \rangle \in \langle I \rangle$, the set $A_{\langle \alpha, \beta \rangle} = \{ x \in S : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta \}$ is called a cut set of A .

Definition 3 [6] Let $\lambda, \mu \in (0, 1]$ and $\lambda < \mu$.

An intuitionistic fuzzy set A in X is said to be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X if the following are satisfied:

$$(IF_1) \quad \mu_A(0) \vee \lambda \geq \mu_A(x) \wedge \mu,$$

$$(IF_2) \quad \nu_A(0) \wedge \mu \leq \nu_A(x) \vee \lambda,$$

$$(IF_3) \quad \mu_A(x) \vee \lambda \geq \mu_A(x * y) \wedge \mu_A(y) \wedge \mu,$$

$$(IF_4) \quad \nu_A(x) \wedge \mu \leq \nu_A(x * y) \vee \nu_A(y) \vee \lambda,$$

for all $x, y \in X$.

Proposition 1 [6] Let A be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X . If $x \leq y$ holds in X , then $\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \mu, \nu_A(x) \wedge \mu \leq \nu_A(y) \vee \lambda$.

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Proposition 2 [6] Let A be an intuitionistic fuzzy ideal with thresholds (λ, μ) of X . If the inequality $x * y \leq z$ holds in X , then for all $x, y, z \in X$,

$$\mu_A(x) \vee \lambda \geq \mu_A(y) \wedge \mu_A(z) \wedge \mu,$$

$$\nu_A(x) \wedge \mu \leq \nu_A(y) \vee \nu_A(z) \vee \lambda.$$

III. INTUITIONISTIC FUZZY IMPLICATIVE IDEALS WITH THRESHOLDS (λ, μ) OF BCI- ALGEBRAS

Definition 4 Let $\lambda, \mu \in (0, 1]$ and $\lambda < \mu$.

An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of X if it satisfies (IF_1) , (IF_2) and

$$(IF_5) \mu_A \left(x * \left((y * (y * x)) * (0 * (0 * (x * y))) \right) \right) \vee \lambda$$

$$\geq \mu_A \left(\left(((x * y) * y) * (0 * y) \right) * z \right) \wedge \mu_A(z) \wedge \mu,$$

$$(IF_6) \nu_A \left(x * \left((y * (y * x)) * (0 * (0 * (x * y))) \right) \right) \wedge \mu$$

$$\leq \nu_A \left(\left(((x * y) * y) * (0 * y) \right) * z \right) \vee \nu_A(z) \vee \lambda.$$

Proposition 3 Any intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of X is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X , but the converse does not hold.

Proof. Assume that A is an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of X and put $y = 0$ in (IF_5) and (IF_6) , we get

$$\mu_A(x) \vee \lambda = \mu_A \left(x * \left((0 * (0 * x)) * (0 * (0 * (x * 0))) \right) \right) \vee \lambda$$

$$\geq \mu_A \left(\left(((x * 0) * 0) * (0 * 0) \right) * z \right) \wedge \mu_A(z) \wedge \mu$$

$$\geq \mu_A(x * z) \wedge \mu_A(z) \wedge \mu,$$

$$\nu_A(x) \wedge \mu = \nu_A \left(x * \left((0 * (0 * x)) * (0 * (0 * (x * 0))) \right) \right) \wedge \mu$$

$$\leq \nu_A \left(\left(((x * 0) * 0) * (0 * 0) \right) * z \right) \vee \nu_A(z) \vee \lambda$$

$$= \nu_A(x * z) \vee \nu_A(z) \vee \lambda.$$

This means that A satisfies (IF_3) and (IF_4) . Combining (IF_1) and (IF_2) , A is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X .

To show the last half part, we see the following example.

Example 1 Let $X = \{0, 1, 2\}$ with Cayley table given by

TABLE I
RESULT OF COMPUTATION

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Define $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \}$ where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ by $\mu_A(0) = 2/3$, $\mu_A(1) = \mu_A(2) = 1/3$,

$\nu_A(0) = 1/4$, $\nu_A(1) = \nu_A(2) = 1/2$. Let $\lambda = 1/8$ and $\mu = 3/4$. By routine calculations give that A is an intuitionistic fuzzy ideal with thresholds (λ, μ) of X . But it is not an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) as

$$\mu_A \left(1 * \left((2 * (2 * 1)) * (0 * (0 * (1 * 2))) \right) \right) \vee \lambda = \mu_A(1)$$

$$< \mu_A(0) = \mu_A \left(\left(((1 * 2) * 2) * (0 * 2) \right) * 0 \right) \wedge \mu_A(0) \wedge \mu.$$

The characterization of intuitionistic fuzzy implicative ideals with thresholds (λ, μ) of X are given by the following proposition.

Proposition 4 An intuitionistic fuzzy set A of X is an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of X if and only if, for all $\alpha, \beta \in (\lambda, \mu]$, $A_{\langle \alpha, \beta \rangle}$ is either empty or an implicative ideal of X .

Proof. Let A be an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of X and $A_{\langle \alpha, \beta \rangle} \neq \emptyset$ for some $\alpha, \beta \in (\lambda, \mu]$. It is clear that $0 \in A_{\langle \alpha, \beta \rangle}$. Let

$$\left(((x * y) * y) * (0 * y) \right) * z \in A_{\langle \alpha, \beta \rangle} \text{ and } z \in A_{\langle \alpha, \beta \rangle}, \text{ then}$$

$$\mu_A \left(\left(((x * y) * y) * (0 * y) \right) * z \right) \geq \alpha, \mu_A(z) \geq \alpha,$$

$$\nu_A \left(\left(((x * y) * y) * (0 * y) \right) * z \right) \leq \beta, \nu_A(z) \leq \beta.$$

It follows from (IF_5) and (IF_6) ,

$$\mu_A \left(x * \left((y * (y * x)) * (0 * (0 * (x * y))) \right) \right) \vee \lambda$$

$$\geq \mu_A \left(\left(((x * y) * y) * (0 * y) \right) * z \right) \wedge \mu_A(z) \wedge \mu \geq \alpha,$$

$$\nu_A \left(x * \left((y * (y * x)) * (0 * (0 * (x * y))) \right) \right) \wedge \mu$$

$$\leq \nu_A \left(\left(((x * y) * y) * (0 * y) \right) * z \right) \vee \nu_A(z) \vee \lambda \leq \beta.$$

Namely, $\mu_A \left(x * \left((y * (y * x)) * (0 * (0 * (x * y))) \right) \right) \geq \alpha$,

$$\nu_A \left(x * \left((y * (y * x)) * (0 * (0 * (x * y))) \right) \right) \leq \beta \text{ and}$$

$$x * \left((y * (y * x)) * (0 * (0 * (x * y))) \right) \in A_{\langle \alpha, \beta \rangle}.$$

This shows that $A_{\langle \alpha, \beta \rangle}$ is an implicative ideal of X .

Conversely, suppose that for each $\alpha, \beta \in (\lambda, \mu]$, $A_{\langle \alpha, \beta \rangle}$ is either empty or an implicative ideal of X . For any $x \in X$, let $\alpha = \mu_A(x) \wedge \mu, \beta = \nu_A(x) \vee \lambda$. Then $\mu_A(x) \geq \alpha, \nu_A(x) \leq \beta$, hence $x \in A_{\langle \alpha, \beta \rangle}$ and $A_{\langle \alpha, \beta \rangle}$ is an implicative ideal of X , therefore $0 \in A_{\langle \alpha, \beta \rangle}$, i.e., $\mu_A(0) \geq \alpha$, and $\nu_A(0) \leq \beta$. We get $\mu_A(0) \vee \lambda \geq \mu_A(0) \geq \alpha = \mu_A(x) \wedge \mu$,

$$\nu_A(0) \wedge \mu \leq \nu_A(0) \leq \beta = \nu_A(x) \vee \lambda,$$

$$\text{i.e., } \mu_A(0) \vee \lambda \geq \mu_A(x) \wedge \mu \text{ and } \nu_A(0) \wedge \mu \leq \nu_A(x) \vee \lambda,$$

for all $x \in X$.

Now we only need to show that A satisfies (IF_5) and (IF_6) .

Let

$$\alpha = \mu_A(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \wedge \mu_A(z) \wedge \mu,$$

$$\beta = \nu_A(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \vee \nu_A(z) \vee \lambda.$$

Then

$$\mu_A(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \geq \alpha, \mu_A(z) \geq \alpha,$$

$$\nu_A(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \leq \beta, \nu_A(z) \leq \beta.$$

Hence $\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \in A_{(\alpha, \beta)}$ and $z \in A_{(\alpha, \beta)}$. Since

$A_{(\alpha, \beta)}$ is an implicative ideal of X , thus

$$x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right) \in A_{(\alpha, \beta)}, \text{ i.e.,}$$

$$\mu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \geq \alpha,$$

$$\nu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \leq \beta.$$

We get

$$\mu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \vee \lambda$$

$$\geq \mu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right)$$

$$\geq \alpha = \mu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \wedge \mu_A(z) \wedge \mu,$$

$$\nu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \wedge \mu$$

$$\leq \nu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right)$$

$$\leq \beta = \nu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \vee \nu_A(z) \vee \lambda.$$

This means that A satisfies (IF_5) and (IF_6) . Hence, A is an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of X .

Proposition 5 Let J be an implicative ideal of X . Then there exists an intuitionistic fuzzy implicative ideal A with thresholds (λ, μ) of X such that $A_{(\alpha, \beta)} = J$ for some

$$\alpha, \beta \in (\lambda, \mu].$$

Proof. Define $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \}$ by

$$\mu_A(x) = \begin{cases} \alpha & \text{if } x \in J, \\ \lambda & \text{if } x \notin J, \end{cases}$$

$$\nu_A(x) = \begin{cases} \beta & \text{if } x \in J, \\ \mu & \text{if } x \notin J, \end{cases}$$

where α, β are two fixed numbers in $(\lambda, \mu]$.

Since J is an implicative ideal of X ,

if $\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \in J$ and $z \in J$, then

$$x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right) \in J.$$

Hence

$$\mu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) = \mu_A(z)$$

$$= \mu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) = \alpha,$$

$$\nu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) = \nu_A(z)$$

$$= \nu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) = \beta,$$

thus

$$\mu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \vee \lambda$$

$$\geq \mu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \wedge \mu_A(z) \wedge \mu,$$

$$\nu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \wedge \mu$$

$$\leq \nu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \vee \nu_A(z) \vee \lambda.$$

If at least one of $\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \in$ and z is not in J ,

then at least one of $\mu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right)$ and $\mu_A(z)$ is λ ,

and at least one of $\nu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right)$ and $\nu_A(z)$ is μ .

Therefore,

$$\mu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \vee \lambda$$

$$\geq \mu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \wedge \mu_A(z) \wedge \mu,$$

$$\nu_A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right)\right) \wedge \mu$$

$$\leq \nu_A\left(\left(\left(\left((x * y) * y\right) * (0 * y)\right) * z\right) \vee \nu_A(z) \vee \lambda.$$

This means that A satisfies (IF_5) and (IF_6) . Since $0 \in J$,

$$\mu_A(0) \vee \lambda = \alpha \geq \mu_A(x) \wedge \mu, \nu_A(0) \wedge \mu = \beta \leq \nu_A(x) \vee \lambda,$$

for all $x \in X$ and so A satisfies (IF_1) and (IF_2) . Thus, A is an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of X . It is clear that $A_{(\alpha, \beta)} = J$.

Definition 5 [6] Let $\lambda, \mu \in (0, 1]$ and $\lambda < \mu$.

An intuitionistic fuzzy ideal A with thresholds (λ, μ) in X is said to be an intuitionistic fuzzy closed ideal with thresholds (λ, μ) of X if the following are satisfied:

$$\mu_A(0 * x) \vee \lambda \geq \mu_A(x) \wedge \mu,$$

$$\nu_A(0 * x) \wedge \mu \leq \nu_A(x) \vee \lambda,$$

for all $x \in X$.

Proposition 6 Let A be an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of X . If A is an intuitionistic fuzzy closed ideal with thresholds (λ, μ) of X , then for all $x, y \in X$,

$$\mu_A(x*(y*(y*x))) \vee \lambda \geq \mu_A(((x*y)*y)*(0*y)) \wedge \mu,$$

$$\nu_A(x*(y*(y*x))) \wedge \mu \leq \nu_A(((x*y)*y)*(0*y)) \vee \lambda.$$

Proof. Assume that A is both an intuitionistic fuzzy implicative ideal and an intuitionistic fuzzy closed ideal with thresholds (λ, μ) of X . Substituting 0 for z in (IF_5) and (IF_6) , we get

$$\mu_A(x*((y*(y*x))*(0*(0*(x*y)))) \vee \lambda$$

$$\geq \mu_A(((x*y)*y)*(0*y)) \wedge \mu,$$

$$\nu_A(x*((y*(y*x))*(0*(0*(x*y)))) \wedge \mu$$

$$\leq \nu_A(((x*y)*y)*(0*y)) \vee \lambda.$$

By Definition 5, we have

$$\mu_A(0*((x*y)*y)*(0*y)) \vee \lambda$$

$$\geq \mu_A(((x*y)*y)*(0*y)) \wedge \mu,$$

$$\nu_A(0*((x*y)*y)*(0*y)) \wedge \mu$$

$$\leq \nu_A(((x*y)*y)*(0*y)) \vee \lambda.$$

Since

$$(x*(y*(y*x)))*((x*(y*(y*x))*(0*(0*(x*y))))$$

$$\leq ((y*(y*x))*(0*(0*(x*y))))*(y*(y*x))$$

$$= 0*(0*(0*(x*y)))$$

$$= 0*(x*y)$$

and

$$0*((x*y)*y)*(0*y)$$

$$= ((0*(x*y))*(0*y))*(0*(0*y))$$

$$= ((0*(0*(0*y)))*(x*y))*(0*y)$$

$$= ((0*y)*(x*y))*(0*y)$$

$$= 0*(x*y),$$

we get

$$(x*(y*(y*x)))*((x*(y*(y*x))*(0*(0*(x*y))))$$

$$\leq 0*((x*y)*y)*(0*y).$$

by Proposition 2, we obtain

$$\mu_A(x*(y*(y*x))) \vee \lambda$$

$$= (\mu_A(x*(y*(y*x))) \vee \lambda) \vee \lambda$$

$$\geq (\mu_A(x*((y*(y*x))*(0*(0*(x*y))))$$

$$\wedge \mu_A(0*((x*y)*y)*(0*y))) \vee \lambda$$

$$= (\mu_A(x*((y*(y*x))*(0*(0*(x*y)))) \vee \lambda)$$

$$\wedge (\mu_A(0*((x*y)*y)*(0*y)) \vee \lambda) \wedge (\mu \vee \lambda)$$

$$\geq (\mu_A(((x*y)*y)*(0*y)) \wedge \mu)$$

$$\wedge (\mu_A(((x*y)*y)*(0*y)) \wedge \mu) \wedge \mu$$

$$= \mu_A(((x*y)*y)*(0*y)) \wedge \mu,$$

$$\nu_A(x*(y*(y*x))) \wedge \mu$$

$$= (\nu_A(x*(y*(y*x))) \wedge \mu) \wedge \mu$$

$$\leq (\nu_A(x*((y*(y*x))*(0*(0*(x*y))))$$

$$\vee \nu_A(0*((x*y)*y)*(0*y))) \vee \lambda \wedge \mu$$

$$= (\nu_A(x*((y*(y*x))*(0*(0*(x*y)))) \wedge \mu)$$

$$\vee (\nu_A(0*((x*y)*y)*(0*y)) \wedge \mu) \vee (\lambda \wedge \mu)$$

$$\leq (\nu_A(((x*y)*y)*(0*y)) \vee \lambda)$$

$$\vee (\nu_A(((x*y)*y)*(0*y)) \vee \lambda) \vee \lambda$$

$$= \nu_A(((x*y)*y)*(0*y)) \vee \lambda.$$

Namely,

$$\mu_A(x*(y*(y*x))) \vee \lambda \geq \mu_A(((x*y)*y)*(0*y)) \wedge \mu,$$

$$\nu_A(x*(y*(y*x))) \wedge \mu \leq \nu_A(((x*y)*y)*(0*y)) \vee \lambda.$$

Definition 6 Let S be any set. If

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \}, B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle : x \in S \}$$

be any two intuitionistic fuzzy subsets of S , then

$$A \cap B = \{ \langle x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x) \rangle : x \in S \}$$

$$= \{ \langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle : x \in S \}$$

Proposition 7 Let A and B be two intuitionistic fuzzy implicative ideals with thresholds (λ, μ) of X . Then $A \cap B$ is also an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of X .

Proof. For all $x, y, z \in X$, by Definition 4, we have

$$\mu_{A \cap B}(0) \vee \lambda = (\mu_A(0) \wedge \mu_B(0)) \vee \lambda$$

$$= (\mu_A(0) \vee \lambda) \wedge (\mu_B(0) \vee \lambda)$$

$$\geq (\mu_A(x) \wedge \mu) \wedge (\mu_B(x) \wedge \mu)$$

$$= (\mu_A(x) \wedge \mu_B(x)) \wedge \mu$$

$$= \mu_{A \cap B}(x) \wedge \mu,$$

$$\nu_{A \cap B}(0) \wedge \mu = (\nu_A(0) \vee \nu_B(0)) \wedge \mu$$

$$= (\nu_A(0) \wedge \mu) \vee (\nu_B(0) \wedge \mu)$$

$$\leq (\nu_A(x) \vee \lambda) \vee (\nu_B(x) \vee \lambda)$$

$$= (\nu_A(x) \vee \nu_B(x)) \vee \lambda$$

$$\begin{aligned}
 &= \nu_{A \cap B}(x) \vee \lambda, \\
 \mu_{A \cap B} &\left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \vee \lambda \\
 &= \left(\mu_A \left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \right) \\
 &\quad \wedge \mu_B \left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \vee \lambda \\
 &= \left(\mu_A \left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \vee \lambda \right) \\
 &\quad \wedge \left(\mu_B \left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \vee \lambda \right) \\
 &\geq \left(\mu_A \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \wedge \mu_A(z) \wedge \mu \right) \\
 &\quad \wedge \left(\mu_B \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \wedge \mu_B(z) \wedge \mu \right) \\
 &= \left(\mu_A \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \right) \\
 &\quad \wedge \mu_B \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \wedge \left(\mu_A(z) \wedge \mu_B(z) \right) \wedge \mu \\
 &= \mu_{A \cap B} \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \wedge \mu_{A \cap B}(z) \wedge \mu. \\
 \nu_{A \cap B} &\left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \wedge \mu \\
 &= \left(\nu_A \left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \right) \\
 &\quad \vee \nu_B \left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \wedge \mu \\
 &= \left(\nu_A \left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \wedge \mu \right) \\
 &\quad \vee \left(\nu_B \left(x * \left(\left(y * (y * x) \right) * \left(0 * \left(0 * (x * y) \right) \right) \right) \right) \wedge \mu \right) \\
 &\leq \left(\nu_A \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \vee \nu_A(z) \vee \lambda \right) \\
 &\quad \vee \left(\nu_B \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \vee \nu_B(z) \vee \lambda \right) \\
 &= \left(\nu_A \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \right) \\
 &\quad \vee \nu_B \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \vee \left(\nu_A(z) \vee \nu_B(z) \right) \vee \lambda \\
 &= \nu_{A \cap B} \left(\left(\left((x * y) * y \right) * \left(0 * y \right) \right) * z \right) \vee \nu_{A \cap B}(z) \vee \lambda.
 \end{aligned}$$

Hence $A \cap B$ is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of X .

Definition 7 Let A and B be two intuitionistic fuzzy sets of a set X . The Cartesian product of A and B is defined by $A \times B = \{ \langle \mu_{A \times B}(x, y), \nu_{A \times B}(x, y) \rangle : x, y \in X \}$ where

$$\mu_{A \times B}(x, y) = \mu_A(x) \wedge \mu_B(y), \nu_{A \times B}(x, y) = \nu_A(x) \vee \nu_B(y).$$

Proposition 8 Let A and B be two intuitionistic fuzzy positive implicative ideals with thresholds (λ, μ) of X . Then $A \times B$ is also an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of $X \times X$.

Proof. For all $(x, y) \in X \times X$, by Definition 4, we get

$$\mu_{A \times B}(0, 0) \vee \lambda = (\mu_A(0) \wedge \mu_B(0)) \vee \lambda$$

$$\begin{aligned}
 &= (\mu_A(0) \vee \lambda) \wedge (\mu_B(0) \vee \lambda) \\
 &\geq (\mu_A(x) \wedge \mu) \wedge (\mu_B(y) \wedge \mu) \\
 &= \mu_{A \times B}(x, y) \wedge \mu,
 \end{aligned}$$

for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have

$$\begin{aligned}
 &\mu_{A \times B} \left(x_1 * \left(\left(y_1 * (y_1 * x_1) \right) * \left(0 * \left(0 * (x_1 * y_1) \right) \right) \right) \right), \\
 &\quad x_2 * \left(\left(y_2 * (y_2 * x_2) \right) * \left(0 * \left(0 * (x_2 * y_2) \right) \right) \right) \vee \lambda \\
 &= \left(\mu_A \left(x_1 * \left(\left(y_1 * (y_1 * x_1) \right) * \left(0 * \left(0 * (x_1 * y_1) \right) \right) \right) \right) \right) \\
 &\quad \wedge \mu_B \left(x_2 * \left(\left(y_2 * (y_2 * x_2) \right) * \left(0 * \left(0 * (x_2 * y_2) \right) \right) \right) \right) \vee \lambda \\
 &= \left(\mu_A \left(x_1 * \left(\left(y_1 * (y_1 * x_1) \right) * \left(0 * \left(0 * (x_1 * y_1) \right) \right) \right) \right) \vee \lambda \right) \\
 &\quad \wedge \left(\mu_B \left(x_2 * \left(\left(y_2 * (y_2 * x_2) \right) * \left(0 * \left(0 * (x_2 * y_2) \right) \right) \right) \right) \vee \lambda \right) \\
 &\geq \left(\mu_A \left(\left(\left((x_1 * y_1) * y_1 \right) * \left(0 * y_1 \right) \right) * z_1 \right) \wedge \mu_A(z_1) \wedge \mu \right) \\
 &\quad \wedge \left(\mu_B \left(\left(\left((x_2 * y_2) * y_2 \right) * \left(0 * y_2 \right) \right) * z_2 \right) \wedge \mu_B(z_2) \wedge \mu \right) \\
 &= \left(\mu_A \left(\left(\left((x_1 * y_1) * y_1 \right) * \left(0 * y_1 \right) \right) * z_1 \right) \right) \\
 &\quad \wedge \mu_B \left(\left(\left((x_2 * y_2) * y_2 \right) * \left(0 * y_2 \right) \right) * z_2 \right) \wedge \left(\mu_A(z_1) \wedge \mu_B(z_2) \right) \wedge \mu \\
 &= \mu_{A \times B} \left(\left(\left((x_1 * y_1) * y_1 \right) * \left(0 * y_1 \right) \right) * z_1 \right), \\
 &\quad \left(\left(\left((x_2 * y_2) * y_2 \right) * \left(0 * y_2 \right) \right) * z_2 \right) \wedge \mu_{A \times B}(z_1, z_2) \wedge \mu.
 \end{aligned}$$

Similarly it can be proved that

$$\nu_{A \times B}(0, 0) \wedge \mu \leq \nu_{A \times B}(x, y) \vee \lambda,$$

$$\begin{aligned}
 &\nu_{A \times B} \left(x_1 * \left(\left(y_1 * (y_1 * x_1) \right) * \left(0 * \left(0 * (x_1 * y_1) \right) \right) \right) \right), \\
 &\quad x_2 * \left(\left(y_2 * (y_2 * x_2) \right) * \left(0 * \left(0 * (x_2 * y_2) \right) \right) \right) \wedge \mu \\
 &\leq \nu_{A \times B} \left(\left(\left((x_1 * y_1) * y_1 \right) * \left(0 * y_1 \right) \right) * z_1 \right), \\
 &\quad \left(\left(\left((x_2 * y_2) * y_2 \right) * \left(0 * y_2 \right) \right) * z_2 \right) \vee \nu_{A \times B}(z_1, z_2) \vee \lambda.
 \end{aligned}$$

Hence $A \times B$ is an intuitionistic fuzzy implicative ideal with thresholds (λ, μ) of $X \times X$.

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