Weighted Composition Operators Acting between Kind of Weighted Bergman-Type Spaces and the Bers-Type Space

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Abstract—In this paper, we study the boundedness and compactness of the weighted composition operator $W_{u,\phi}$, which is induced by an holomorphic function $u$ and holomorphic self-map $\phi$, acting between the $N_K$-space and the Bers-type space $H_\alpha^{\infty}$ on the unit disk.

Keywords—Weighted composition operators, $N_K$-space, Bers-type space.

I. INTRODUCTION

Let $D = \{ z : |z| < 1 \}$ be the unit disk in the complex plane, $\partial D$ its boundary. $H(D)$ denotes the class of all analytic functions on $D$, while $dA(z)$ denotes the Lebesgue measure on the plane, normalized so that $A(D) = 1$. For each $a \in D$, the Green’s function with logarithmic singularity at $a \in D$ is denoted by $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$ is a Möbius transformations of $D$. The pseudo-hyperbolic disk $D(a, r)$ is defined by

$$ D(a, r) = \{ z \in D : |\varphi_a(z)| < r \}. $$

We will frequently use the following easily verified equality:

$$(1 - |\varphi_a(z)|^2)^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|^2}. $$

For $p \in (0, \infty)$ and $-1 < \alpha < \infty$, the Bers-type spaces $H_\alpha^{\infty}$ consists of all $f \in H(D)$ such that

$$ \|f\|_{\alpha} = \sup_{z \in D} |f(z)|(1 - |z|^2)^\alpha < \infty, $$

and $H_{\alpha,0}^{\infty}$ consists of all $f \in H(D)$ such that

$$ \|f\|_{\alpha,0} = \lim_{|z| \to 1} |f(z)|(1 - |z|^2)^\alpha = 0. $$

For more information about several studied on Bers-type spaces we refer to [3], [12].

For $0 < \alpha < \infty$ the $\alpha$-Bloch space $B_\alpha$ consists of all $f \in H(D)$ such that

$$ \|f\|_{B_\alpha} = \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty. $$

Moreover, $f \in B_\alpha^0$ if

$$ \|f\|_{B_\alpha^0} = \lim_{|z| \to 1} |f'(z)|(1 - |z|^2)^\alpha = 0. $$

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The space $B_1^1$ is called the Bloch space $B$ (see [11]). For each $\alpha > 0$, we know that $H_\alpha^{\infty} = B^{1+\alpha}$ and $H_{\alpha,0}^{\infty} = B_0^{1+\alpha}$ (see [13], Proposition 7).

El-Sayed Ahmed and Bakhit in [4] introduced the $N_K$ spaces (with the right continuous and nondecreasing function $K : [0, \infty) \to [0, \infty)$) consists of $f \in H(D)$ such that

$$ \|f\|_{N_K}^2 = \sup_{a \in D} \int_D |f(z)|^2 K(g(z, a))dA(z) < \infty. $$

If

$$ \lim_{|t| \to 1} \int_D |f(z)|^2 K(g(z, a))dA(z) = 0, $$

then $f$ is said to belong to $N_{K,0}$. For $K(t) = 1$ it gives the Bergman space. If $N_K$ consists of just the constant functions, we say that it is trivial. Clearly, if $K(t) = t^p$, then $N_K = N_{p}$; since $K(t) = |\varphi_a(z)|^2$. The $N_{K}$-space was introduced by Palmberg in [8]. Finally, when $K(t) = t$, $N_K$ coincides $N_1$, the $N_1$-space was introduced in [7].

From a change of variable we see that the coordinate $z$ belongs to $N_K$ space if and only if

$$ \sup_{a \in D} \int_D \frac{(1 - |a|^2)^2}{|1 - \overline{a}z|^4} K \left( \log \frac{1}{|z|} \right) dA(z) < \infty. $$

Simplifying the above integral in polar coordinates, we conclude that $N_K$ space is nontrivial if and only if

$$ \sup_{t \in (0,1]} \int_0^1 \frac{(1 - t)^2}{(1 - tr^2)^2} K \left( \log \frac{1}{r} \right) rdr < \infty. $$

We assume from now that all $K : [0, \infty) \to [0, \infty)$ appear in this paper are right-continuous and nondecreasing function. Moreover, we always assume that condition (1) is satisfied, so that the $N_K$ space we study is not trivial.

Given $u \in H(D)$ and $\phi$ a holomorphic self-map of $D$. The weighted composition operator $W_{u,\phi} : H(D) \to H(D)$ is defined by

$$ W_{u,\phi}(f)(z) = u(z)(f \circ \phi)(z), \quad z \in D. $$

It is obvious that $W_{u,\phi}$ can be regarded as a generalization of the multiplication operator $M_{u}f = u \cdot f$ and composition operator $C_{\phi}f = f \circ \phi$. The behavior of those operators is studied extensively on various spaces of holomorphic functions (see for example [3], [4], [6], [7], [8]). El-Sayed Ahmed and Bakhit in [4] considered the composition operator $C_{\phi}f = f \circ \phi$ on the space $N_K$. They gave complete characterizations for the boundedness and compactness of $C_{\phi} : N_K \to H_\alpha^{\infty}$. However...
the boundedness and compactness of the case $C_\phi : H^\infty_\alpha \to \mathcal{N}_K$ remain to be studied.

In this paper, we will characterize the boundedness and compactness of the case $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ and $W_{u,\phi} : \mathcal{N}_K \to H^\infty_\alpha$. Our situations have not been covered by a recent progress of studies of weighted composition operators. Of course, the results in this paper will also give the characterizations of the case $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ and the case $W_{u,\phi} : \mathcal{N}_K \to H^\infty_\alpha$ is a generalization of the results in [4], [8] and [10]. Furthermore, by the derivative operator $f \to f', Q_K$-spaces (see [9]) are closely related to $\mathcal{N}_K$-spaces and Bloch-type spaces $B^\alpha$ related to $H^\infty_\alpha$.

For a subarc $I \subset \partial D$, let

$$S(I) = \{r \zeta \in D : 1 - |I| < r < 1, \zeta \in I\}.$$

If $|I| \geq 1$ then we set $S(I) = D$. For $0 < p < \infty$, we say that a positive measure $d\mu$ is a $p$-Carleson measure on $D$ if

$$\sup_{I \subset \partial D} \frac{\mu(S(I))}{|I|^p} < \infty.$$ 

Here and henceforth $\sup_{I \subset \partial D}$ indicates the supremum taken over all subarcs $I$ of $\partial D$. Note that $p = 1$ gives the classical Carleson measure (see [1], [2]). A positive measure $d\mu$ is said to be a $1$-Carleson measure on $D$ if

$$\sup_{I \subset \partial D} \int_{S(I)} K\left(\frac{1 - |z|}{|I|}\right) d\mu(z) < \infty.$$ 

Clearly, if $K(t) = t^p$, then $\mu$ is a $K$-Carleson measure on $D$ if and only if $(1 - |z|^2)d\mu$ is a $p$-Carleson measure on $D$.

Pau in [9] proved the following results:

**Lemma 1.** Let $K$ satisfy (1) and $\mu$ be a positive measure. Then

(i) $\mu$ is a $K$-Carleson measure if and only if

$$\sup_{a \in D} \int_D K(1 - |\phi_a(z)|^2) dA(z) < \infty.$$ 

(ii) $\mu$ is a compact $K$-Carleson measure if and only if (2) holds and

$$\lim_{|a| \to 1} \int_D K(1 - |\phi_a(z)|^2) dA(z) = 0.$$ 

**Lemma 2.** Let $K$ satisfy (1) and let $f \in \mathcal{H}(D)$. Then the following are equivalent.

(i) $f \in \mathcal{N}_K$.

(ii) $\sup_{a \in D} \int_D |f \circ \phi_a(z)|^2 K(1 - |z|^2) dA(z) < \infty$.

(iii) $|f(z)|^2 dA(z)$ is a $K$-Carleson measure on $D$.

**Lemma 3.** (Test function in $\mathcal{N}_K$ see [5], Lemma 2.2) Let $K$ satisfy (1). For $w \in D$ we define

$$h_w(z) = \frac{1 - |w|^2}{(1 - \bar{w}z)^2}.$$ 

Then $h_w \in \mathcal{N}_K$ and $\|h_w\|_{\mathcal{N}_K} \leq 1$.

The following lemma proved by Ueki (see [10], Lemma 2):

**Lemma 4.** (Test function in $H^\infty_\alpha$) For each $\alpha \in (0, \infty)$, $\theta \in [0, 2\pi)$, $r \in (0, 1)$ and $w \in D$, we put

$$h_{\theta, r}(w) := \sum_{k=0}^{\infty} 2^k \alpha (r e^{i\theta})^{2^k} w^{2^k}.$$ 

Then $h_{\theta, r} \in H^\infty_\alpha$ and $\|h_{\theta, r}\|_{H^\infty_\alpha} \leq 1$. In particular, $h_{\theta, r} \in H^\infty_0$ if $r \in (0, 1)$.

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \leq C f\|_X$ for all maps $f \in X$. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in $X$ to sets in $Y$ which have compact closure. For Banach spaces $X$ and $Y$ of $H(\Delta)$, $T$ is compact from $X$ to $Y$ if and only if for each bounded sequence $\{x_n\} \in X$, the sequence $\{T x_n\} \in Y$ contains a subsequence converging to some limit in $Y$.

Two quantities $A_f$ and $B_f$, both depending on an $f \in \mathcal{H}(D)$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant $C$ not depending on $f$ such that for every analytic function $f$ on $D$ we have

$$\frac{1}{C} B_f \leq A_f \leq C B_f.$$ 

If the quantities $A_f$ and $B_f$, are equivalent, then in particular we have $A_f \approx C$ if and only if $B_f \approx C$. As usual, the letter $C$ will denote a positive constant, possibly different on each occurrence.

II. WEIGHTED COMPOSITION OPERATORS FROM $H^\infty_\alpha$ INTO $\mathcal{N}_K$ SPACES

In this section, we characterize the boundedness and compactness of weighted composition operators $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$. First, in the following result, we describe the boundedness of $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$.

**Theorem 1.** Let $K : [0, \infty) \to [0, \infty)$ be a nondecreasing function and $\phi$ be a holomorphic self-map of $D$. For $\alpha \in (0, \infty)$ and $u \in \mathcal{H}(D)$, then the following are equivalent:

(i) $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ is a bounded operator.

(ii) $u$ and $\phi$ satisfy:

$$\sup_{a \in D} \int_D (1 - |\phi(z)|^2)^{\alpha a} K(g(z, a)) dA(z) < \infty.$$ 

(iii) $u$ and $\phi$ satisfy:

$$\sup_{I \subset \partial D} \int_{S(I)} (1 - |\phi(z)|^2)^{\alpha a} K(1 - |z|^2) dA(z) < \infty.$$ 

**Proof.** (ii) $\Rightarrow$ (i). We assume that condition (3) holds and let

$$\sup_{a \in D} \int_D (1 - |\phi(z)|^2)^{\alpha a} K(g(z, a)) dA(z) < C,$$ 

where $C$ is a positive constant. If $f \in H^\infty_\alpha$, then for all $a \in D$, we have

$$\|W_{u,\phi}(f)\|_{\mathcal{N}_K} = \sup_{a \in D} \int_D |u(z)|^2 |f(\phi(z))|^2 K(g(z, a)) dA(z) \leq \|f\|_{H^\infty_\alpha}^2 \sup_{a \in D} \int_D |u(z)|^2 K(g(z, a)) dA(z) \leq C \|f\|_{H^\infty_\alpha}^2.$$
(i) $\Rightarrow$ (ii). Suppose that $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ is bounded, then
\[ \|W_{u,\phi}(f)\|_{\mathcal{N}_K} \leq \|f\|_{H^\infty_\alpha}. \]

For each $\alpha \in (0, \infty), \theta \in [0, 2\pi)$ we set the test function $h_\theta = h_{\theta, 1}$ which is defined in Lemma 4 with $w = \phi(z_0)$. Fix $w \in D$, by Fubini's theorem we have
\[ 1 \geq \int_0^{2\pi} \|W_{u,\phi}(h_\theta)\|_{\mathcal{N}_K} \frac{d\theta}{2\pi} \geq \int_0^{2\pi} |u(z)|^2 K(g(z, a)) \int_0^{2\pi} |h_\theta(\phi(z))|^2 \frac{d\theta}{2\pi} dA(z). \]

By Parseval's formula as in [10], when $|\phi(z)| > \frac{1}{\sqrt{2}}$, we have
\[ \int_0^{2\pi} |h_\theta(\phi(z))|^2 \frac{d\theta}{2\pi} \geq \frac{1}{(1 - |\phi(z)|^2)^{2\alpha}}. \]

Hence we obtain
\[ \int_{\partial D} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K[g(z, a)] dA(z) \leq 1, \quad (5) \]
for any $a \in D$, where $D_1 = \{ z \in D : |\phi(z)| > \frac{1}{\sqrt{2}} \}$. By noting that $u \in \mathcal{N}_K$, for any $a \in D$, we have
\[ \int_{z \in \partial D} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K[g(z, a)] dA(z) \leq C \|u\|_{\mathcal{N}_K}. \]

Inequalities (5) and (6) show that the condition (3) is true. (iii) $\Rightarrow$ (i). For every $f \in H^\infty_\alpha$ it follows that
\[ \sup_{I \subset \partial D} \int_{S(I)} |u(z)|^2 |f(\phi(z))|^2 K(1 - |z|) dA(z) \leq \|f\|_{H^\infty_\alpha}^2 \sup_{I \subset \partial D} \int_{S(I)} |u(z)|^2 K(1 - |z|) dA(z). \]

Combining this with condition (4), we see that
\[ \|f\|_{H^\infty_\alpha}^2 \sup_{I \subset \partial D} \int_{S(I)} |u(z)|^2 |f(\phi(z))|^2 K(1 - |z|) dA(z) \leq \|f\|_{H^\infty_\alpha}^2 \sup_{I \subset \partial D} \int_{S(I)} |u(z)|^2 K(g(z, a)) dA(z) \leq C \|f\|_{H^\infty_\alpha}^2, \]
and so $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ is bounded. (i) $\Rightarrow$ (iii). Assume that $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ is bounded. Fix an arc $I \subset \partial D$, again we consider the test function $h_\theta, \theta \in [0, 2\pi)$. By Lemma 1 and Lemma 4, we have
\[ \int_{S(I) \cap D} |u(z)|^2 \frac{dA(z)}{(1 - |\phi(z)|^2)^{2\alpha}} K((1 - |z|)/|I|) dA(z) \leq 1. \]

Since $u \in \mathcal{N}_K$ by the boundedness of $W_{u,\phi}$, it follows from Lemma 1 that $|u(z)|^2 dA(z)$ is a $K$-Carleson measure and
\[ \sup_{I \subset \partial D} \int_{S(I)} |u(z)|^2 dA(z) \leq \|u\|_{\mathcal{N}_K}^2. \]

Then we have
\[ \int_{S(I) \cap \{|\phi(z)| \leq \frac{1}{\sqrt{2}}\}} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K\left(\frac{1 - |z|}{|I|}\right) dA(z) \leq \|u\|_{\mathcal{N}_K}^2. \]

Hence, we obtain the condition (4) and we accomplish the proof.

Under the same assumption in Theorem 1 we obtain the following theorem.

**Theorem 2.** Let $K : [0, \infty) \to [0, \infty)$ be a nondecreasing function and $\phi$ be a holomorphic self-map of $D$. For $\alpha \in (0, \infty)$ and $u \in H(D)$, then the following are equivalent

(i) $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ is a compact operator.

(ii) $u$ and $\phi$ satisfy:
\[ \lim_{\rho \to 1} \sup_{a \in D} \int_{D_\rho} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(1 - |z|) dA(z) = 0. \]

(iii) $u$ and $\phi$ satisfy:
\[ \lim_{\rho \to 1} \sup_{I \subset \partial D} \int_{S(I) \cap D_\rho} \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(1 - |z|) dA(z) = 0, \]
where $D_\rho = \{ z \in D : |\phi(z)| > \rho \}$.

**Theorem 3.** Suppose $\alpha \in (0, \infty), u \in H(D)$ and let $K : [0, \infty) \to [0, \infty)$ be a nondecreasing function and $\phi$ be a holomorphic self-map of $D$. Then $W_{u,\phi} : H^\infty_\alpha \to \mathcal{N}_K$ is a bounded operator if and only if
\[ \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} dA(z) \text{ is a } K\text{-Carleson measure}. \]

**Proof.** Necessity. By Lemma 1, it suffices to prove that
\[ \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(1 - |\phi(a)|^2) dA(z) < \infty. \]

Since $K$ is nondecreasing and $(1 - t^2) \leq 2 \log \frac{1}{t}$, for $t \in (0, 1]$, we have $1 - |\phi(a)|^2 \leq 2 \log \frac{1}{|\phi(a)|} \leq 2g(z, a)$, for all $z, a \in D$. Using Theorem 1, we have
\[ \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(1 - |\phi(a)|^2) dA(z) \leq \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(2g(z, a)) dA(z) \leq \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(1) dA(z) < \infty. \]

Sufficiency. Assume that
\[ \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} dA(z) \text{ is a } K\text{-Carleson measure}. \]
Then
\[ \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(1 - |\phi(a)|^2) dA(z) < \infty. \]

We obtain that for all $f \in H^\infty_\alpha$,
\[ \sup_{a \in D} \int_D |W_{u,\phi}(f)(z)|^2 K(1 - |\phi(a)|^2) dA(z) \leq \sup_{a \in D} \int_D |u(z)|^2 |f(\phi(z))|^2 K(1 - |\phi(a)|^2) dA(z) \leq \|f\|_{H^\infty_\alpha} \sup_{a \in D} \int_D \frac{|u(z)|^2}{(1 - |\phi(z)|^2)^{2\alpha}} K(1 - |\phi(a)|^2) dA(z) \leq \infty. \]
By Lemma 1, \( W_{u,\phi}(f) \in \mathcal{N}_K \). Thus \( W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K \) is a bounded operator. The proof is completed.

### III. Weighted Composition Operators from \( \mathcal{N}_K \) into \( H_\alpha^\infty \)

In this section, we will consider the operator \( W_{u,\phi} : \mathcal{N}_K \to H_\alpha^\infty \). The case \( u \equiv 1 \) can be found in the work [4] by El-Sayed Ahmed and Bakhit.

**Theorem 4.** Let \( K : [0, \infty) \to [0, \infty) \) be a nondecreasing function and \( \phi \) be a holomorphic self-map of \( D \). For \( \alpha \in (0, \infty) \) and \( u \in H(D) \), then \( W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K \) is a bounded operator if and only if

\[
\sup_{z \in D} \frac{|u(z)|^2(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} < \infty. \tag{7}
\]

**Proof.** We know that \( \mathcal{N}_K \subset H_\alpha^\infty \), for each nondecreasing function \( K : [0, \infty) \to [0, \infty) \) (see [4], Proposition 2.1). First assume that condition (7) holds. Then

\[
\|W_{u,\phi}(f)\|_{H_\alpha^\infty} = \sup_{z \in D} |u(z)| |f(\phi(z))|(1 - |z|^2)^\alpha \\
\leq \|f\|_{H_\alpha^\infty} \sup_{z \in D} \frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} \\
\leq C\|f\|_{\mathcal{N}_K}.
\]

This implies that \( W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K \) is a bounded operator.

Conversely, assume that \( W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K \) is bounded, then

\[
\|W_{u,\phi}(f)\|_{H_\alpha^\infty} \leq \|f\|_{\mathcal{N}_K}.
\]

Fix a point \( z_0 \in D \), and let \( h_w \) be the test function in Lemma 3 with \( w = \phi(z_0) \). Then,

\[
1 \geq \|h_w\|_{\mathcal{N}_K} \geq C_1 \|W_{u,\phi}(h_w)\|_{H_\alpha^\infty} \\
\geq \frac{|u(z_0)|(1 - |w|^2)}{1 - |\phi(z_0)|^2} (1 - |w|^2)^\alpha \\
= \frac{|u(z_0)|(1 - |z_0|^2)^\alpha}{1 - |\phi(z_0)|^2},
\]

where \( C_1 \) is a positive constant. This completes the proof of the theorem.

**Theorem 5.** Let \( K : [0, \infty) \to [0, \infty) \) be a nondecreasing function and \( \phi \) be a holomorphic self-map of \( D \). For \( \alpha \in (0, \infty) \) and \( u \in H(D) \), then \( W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K \) is a compact operator if and only if

\[
\lim_{r \to 1} \sup_{|z| < r} \frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} = 0. \tag{8}
\]

**Proof.** First assume that \( W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K \) is compact and suppose that there exists \( \varepsilon_0 > 0 \) a sequence \( \{z_n\} \subset D \) such that

\[
\frac{|u(z_n)|(1 - |z_n|^2)^\alpha}{1 - |\phi(z_n)|^2} \geq \varepsilon_0,
\]

whenever \( |\phi(z_n)| > 1 - \frac{1}{n} \). Clearly, we can assume that \( w_n = \phi(z_n) \) tends to \( w_0 \in \partial D \) as \( n \to \infty \). Let \( h_w = \frac{(1 - |w_n|^2)}{1 - |\phi(z_n)|^2} \) be the function in Lemma 3. Then \( h_w \to h_w \) with respect to the compact-open topology. Define \( f_n = h_w - h_{w_n} \). By Lemma 3, we have \( \|f_n\|_{\mathcal{N}_K} \leq 1 \) and \( f_n \to 0 \) uniformly on compact subsets of \( D \). Thus, \( f_n \circ \phi \to 0 \) in \( H_\alpha^\infty \) by assumption. But, for \( n \) big enough,

\[
\|W_{u,\phi}(f_n)\|_{H_\alpha^\infty} \geq \frac{|u(z_n)||h_w - h_{w_n}|(1 - |z_n|^2)^\alpha}{1 - |\phi(z_n)|^2} \geq \frac{1}{1 - |\phi(z_n)|^2} \frac{1}{1 - |w_n|^2},
\]

which is a contradiction.

To prove the necessity of (8), we assume that for all \( \varepsilon > 0 \) there exists \( \delta \in (0, 1) \) such that

\[
\frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} < \varepsilon,
\]

whenever \( |\phi(z)| > \delta \). Let \( \{f_n\} \) be a bounded sequence in \( \mathcal{N}_K \) norm which converges to zero on compact subsets of \( D \).

Clearly, we may assume that \( |\phi(z)| > \delta \). Then

\[
\|W_{u,\phi}(f_n)\|_{H_\alpha^\infty} = \sup_{z \in D} |u(z)| \|f_n(\phi(z))|(1 - |z|^2)^\alpha \\
\leq \sup_{z \in D} \frac{|u(z)|(1 - |z|^2)^\alpha}{1 - |\phi(z)|^2} \|f_n(\phi(z))\|(1 - |\phi(z)|^2)^\alpha \\
\leq C \|f_n\|_{H_\alpha^\infty} \leq \varepsilon C \|f_n\|_{\mathcal{N}_K} \leq \varepsilon.
\]

It follows that \( W_{u,\phi} : H_\alpha^\infty \to \mathcal{N}_K \) is a compact operator. This completes the proof of the theorem.

**References**


[10] S.-I. Ueki, Weighted Composition operators acting between the \( \mathcal{N}_p \) -space and the weighted-type space \( H_\alpha^\infty \), Indagationes Mathematicae 23 (2012), 243-255.

