Stability Criteria for Uncertainty Markovian Jumping Parameters of BAM Neural Networks with Leakage and Discrete Delays

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Abstract—In this paper, the problem of stability criteria for Markovian jumping BAM neural networks with leakage and discrete delays has been investigated. Some new sufficient condition are derived based on a novel Lyapunov-Krasovskii functional approach. These new criteria based on delay partitioning idea are proved to be less conservative because free-weighting matrices method and a convex optimization approach are considered. Finally, one numerical example is given to illustrate the usefulness and feasibility of the proposed main results.

Keywords—Stability, Markovian jumping neural networks, Time-varying delays, Linear matrix inequality.

I. INTRODUCTION

Bidirectional associative memory (BAM) neural networks have been extensively studied in recent years due to its wide application in various areas such as image processing, pattern recognition, automatic control, associative memory, optimization problems, and so on. BAM neural network is composed of neurons arranged in two layers: the x-layer and y-layer. The neurons in one layer are fully interconnected to the neurons in the other layer. Now, many sufficient conditions ensuring stability BAM neural networks have been derived, see, for example, [1-19] and references cited therein.

On the other hand, systems with Markovian jumps have been attracting increasing research attention. The Marvokian jump systems have the advantage of modeling the dynamic systems subject to abrupt variation in their structures, such as operating in different points of a nonlinear plant [16]. Recently, there has been a growing interest in the study of neural networks with Markovian jumping parameters [20-28]. In [20], the problem of stochastic stability criteria for BAM neural networks with Markovian jumping parameters are investigated based on partitioning idea. In addition, the authors in [25] discussed the problem of BAM neural networks with constant delays in the leakage term. Moreover, Peng [26], investigated global attractive periodic solution of BAM neural networks with continuously distributed delays in the leakage terms. To the best of our knowledge, the stability analysis for Markovian jumping BAM neural networks with leakage and discrete delays has never been tackled, and such a situation motivates our present study.

In this paper, the stability analysis for Markovian jumping BAM neural networks with leakage and discrete delays is considered. Some new delay-dependent stability criteria for Markovian jumping BAM neural networks with leakage and discrete delays will be proposed by dividing the delay interval into multiple segments, and constructing new Lyapunov-Krasovskii functional. The obtained criterion are less conservative because free-weighting matrices method and a convex optimization approach are considered. Finally, one numerical example is given to illustrate the usefulness and feasibility of the proposed main results.

II. PROBLEM STATEMENT

Consider the following BAM neural networks with leakage and discrete delays:

\[
\begin{align*}
\dot{x}_p(t) &= -Ax_p(t-\sigma) + C\tilde{f}(y_q(t)) + E\tilde{f}(y_q(t-h(t))) + I_p \\
y_q(t) &= -By_q(t-\delta) + Dq(x_p(t)) + F\tilde{g}(y_q(t-\varsigma(t))) + J_q
\end{align*}
\]

(1)

where \(x_p(t) = [x_{p1}(t), x_{p2}(t), \ldots, x_{pn}(t)]^T \in \mathbb{R}^n\) and \(y_q(t) = [y_{q1}(t), y_{q2}(t), \ldots, y_{qn}(t)]^T \in \mathbb{R}^n\) denote the state vectors; \(\tilde{g}(x_p(\cdot)) = [\tilde{g}_1(x_{p1}(\cdot)), \tilde{g}_2(x_{p2}(\cdot)), \ldots, \tilde{g}_n(x_{pn}(\cdot))]^T \in \mathbb{R}^n\) and \(\tilde{f}(y_q(\cdot)) = [\tilde{f}_1(y_{q1}(\cdot)), \tilde{f}_2(y_{q2}(\cdot)), \ldots, \tilde{f}_n(y_{qn}(\cdot))]^T \in \mathbb{R}^n\) are the neuron activation functions; \(A = diag\{a_i\} \in \mathbb{R}^n\) and \(B = diag\{b_i\} \in \mathbb{R}^n\) are positive diagonal matrices; \(C\) and \(D\) are the connection weight matrices, \(E\) and \(F\) are the delayed connection weight matrices; \(I_p = [I_{p1}, I_{p2}, \ldots, I_{pn}]^T \in \mathbb{R}^n\) and \(J_q = [J_{q1}, J_{q2}, \ldots, J_{qn}]^T \in \mathbb{R}^n\) are the constant input vector; \(\varsigma\) and \(\delta\) are the leakage delays satisfying \(\varsigma \geq 0\) and \(\delta \geq 0\), respectively.

The following assumptions are adopted throughout the paper.

Assumption 1: The delay \(h(t)\) and \(\varsigma(t)\) are time-varying continuous functions and satisfies:

\[
0 \leq \varsigma(t) \leq \varsigma, \varsigma(t) \leq \varsigma_D < 1, 0 \leq h(t) \leq h, h(t) \leq h_D < 1
\]

(2)

where \(\varsigma, h, \varsigma_D\) and \(h_D\) are constants.

Assumption 2: Neuron activation function \(g_i(\cdot), f_i(\cdot)\) in (1)
satisfies the following condition:
\[
\begin{align*}
I_{1i}^* & \leq \bar{f}(\alpha) - \tilde{f}(\beta) - \varphi \leq I_{2i}^*, \bar{f}(0) = 0 \\
I_{2i}^* & \leq \tilde{g}(\alpha) - \tilde{g}(\beta) - \varphi \leq I_{2i}^*, \tilde{g}(0) = 0
\end{align*}
\]
for all \(\alpha, \beta \in \mathbb{R}, \alpha \neq \beta, i = 1, \ldots, n\).

Based on this assumption, it can be easily proven that there exists one equilibrium point for (1) by Brouwer’s fixed-point theorem. Let \(x^*_p = [x^*_{p1}, x^*_{p2}, \ldots, x^*_{pn}]^T\), \(y^*_q = [y^*_{q1}, y^*_{q2}, \ldots, y^*_{qn}]^T\) is the equilibrium point of (1) and using the transformation \(x(t) = x^*_p - x^*_p y^*_q = y^*_q - y^*_q\), system (1) can be converted to the following system:
\[
\begin{align*}
\dot{x}(t) &= -Ax(t) + C f(y(t)) + E f(y(t - h(t))) \\
\dot{y}(t) &= -B y(t - \delta) + D g(x(t)) + F g(x(t - c(t)))
\end{align*}
\]
where \(x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T\), \(y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T\), \(g(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t))]^T\), \(f(y(t)) = [f_1(y_1(t)), f_2(y_2(t)), \ldots, f_n(y_n(t))]^T\), \(f_1(y_1(t)) = \bar{f}(y_1(t)) + y^*_q, f_i(y_i(t)) = \bar{f}(y_i(t)) + x^*_p, g_i(x_i(t)) = \tilde{g}(x_i(t) + x^*_p) - \bar{g}(x_i(t)), i = 1, 2, \ldots, n\).

From inequalities (3) and (4), one can obtain that:
\[
\begin{align*}
I_{1i}^* & \leq \bar{f}(\alpha) \leq I_{1i}^*, \bar{f}(0) = 0 \\
I_{2i}^* & \leq \tilde{g}(\alpha) \leq I_{2i}^*, \tilde{g}(0) = 0, i = 1, 2, \ldots, n.
\end{align*}
\]
Given probability space \((\Omega, \mathcal{F}, P)\), where \(\Omega\) is sample space, \(\mathcal{F}\) is \(\sigma\)-algebra of subset of the sample space, and \(P\) is the probability measure defined on \(\Omega\). Let \(\{r(t), t \in [0, +\infty)\}\) be a right-continuous Markovian process on the probability space which takes values in the finite space \(S = \{1, 2, \ldots, N\}\) with generator \(\Pi = (\pi_{ij})_{N \times N}\) given by:
\[
P\{r(t + \Delta t) = j| r(t) = i\} = \left\{ \begin{array}{ll} \pi_{ij}\Delta t + o(\Delta t) & j \neq i \\ 1 + \pi_{ii}\Delta t + o(\Delta t) & j = i \end{array} \right.
\]
with transition rates \(\pi_{ij} \geq 0\) for \(i, j \in S, j \neq i\) and \(\pi_{ii} = -\sum_{j=1, j \neq i}^{N} \pi_{ij}\), where \(\Delta t > 0\) and \(\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0\). Due to the disturbance frequency occurs in many applications, and combining with the discussion above, in this paper, we consider delayed BAM neural networks with uncertainty Markovian jumping parameters described by the following nonlinear differential equations:
\[
\begin{align*}
\dot{x}(t) &= -A(r(t), t)x(t) + C(r(t), t)f(y(t)) \\
&\quad + E(r(t), t)g(x(t - h(t)), t) \\
\dot{y}(t) &= -B(r(t), t)y(t - \delta) + D(r(t), t)g(x(t)) \\
&\quad + F(r(t), t)g(x(t - c(t)), t)
\end{align*}
\]
when \(r(t) = i \in S\) and the matrix functions \(A(r(t), t), B(r(t), t), C(r(t), t), D(r(t), t), E(r(t), t), F(r(t), t), h(r(t), t), c(r(t), t)\) are denoted as \(A_i, B_i, C_i, D_i, E_i, F_i, h_i, c_i\), respectively, and \(h_i(t), c_i(t)\) denote the time-varying delays which satisfy \(\tilde{h}_i(t) \leq h_i(t) \leq \bar{h}_i < 1, \tilde{c}_i(t) \leq c_i(t) \leq \bar{c}_i < 1\).
Theorem 1 For any given scalars $h_i \geq 0, \varsigma_i \geq 0, h_{Di}, \varsigma_{Di}$ and integers $l \geq 1, k \geq 1$, the system (9) with leakage and discrete delays is globally asymptotically stable if there exist symmetric positive definite matrices $P_i, P_{ii}, Q_{ii}, M_j, (j = 1, 2, 3, 4)$. $S_{ij}, (i = 1, \ldots, 6). R_j, (j = 1, \ldots, 8)$ positive diagonal matrices $W_{ij}, (j = 1, \ldots, 6)$ and any matrices $X_i = \begin{bmatrix} X_{i11} & X_{i12} & X_{i13} \\ * & X_{i22} & X_{i23} \\ * & * & X_{i33} \end{bmatrix}, Y_i = \begin{bmatrix} Y_{i11} & Y_{i12} & Y_{i13} \\ * & Y_{i22} & Y_{i23} \\ * & * & Y_{i33} \end{bmatrix}$, $U_i = \begin{bmatrix} U_{i11} & U_{i12} & U_{i13} \\ * & U_{i22} & U_{i23} \\ * & * & U_{i33} \end{bmatrix}, V_i = \begin{bmatrix} V_{i11} & V_{i12} & V_{i13} \\ * & V_{i22} & V_{i23} \\ * & * & V_{i33} \end{bmatrix}$ with appropriate dimensions for any $i = 1, 2, \ldots, N$, such that the following LMIs hold:

\[\begin{align*}
\sum_{i=1}^{N} \pi_{ij} Q_{ik} &< M_k, \ k = 1, 2, 3, 4 \\
\sum_{i=1}^{N} \pi_{ij} S_{ik} &< R_k, \ k = 1, 2, \ldots, 6 \\
X_{i1} &\geq 0 \\
Y_{i1} &\geq 0 \\
U_{i1} &\geq 0 \\
V_{i1} &\geq 0 \\
[X + \Xi + \Xi T R T \ h \Xi T R \ h] &< 0 \\
\Xi &= [\Xi_{nn}], \ \Xi = [\Xi_{nn}], \ (m, n = 1, 2, \ldots, 14) \\
\Xi_{11} &= Q_{11} - (1 - \frac{\pi_{ii}}{h_i}) E_i Q_{11} E_i^T + \tilde{I}_1^T X_{i1} \tilde{I}_1 + \tilde{I}_1^T X_{i1} \tilde{I}_1 \\
\Xi_{12} &= -\tilde{I}_1^T P_{i1} A_i, \ \Xi_{13} = -(1 - \frac{\pi_{ii}}{h_i}) E_i Q_{11} \tilde{I}_1^T \\
\Xi_{14} &= \varsigma_i \tilde{I}_1 X_{i1} - \tilde{I}_1^T X_{i1} + \tilde{I}_1^T X_{i1}^2 \\
\Xi_{15} &= \tilde{I}_1^T L_2 W_{i41}, \ \Xi_{11,12} = \tilde{I}_1^T P_{i1} C_i \\
\Xi_{1,14} &= \tilde{I}_1^T P_{i1} E_i, \ \Xi_{22} = -Q_{3i} \\
\Xi_{33} &= -(1 - \frac{\pi_{ii}}{h_i}) E_i Q_{11} E_i^T - (1 - \frac{\pi_{ii}}{h_i}) S_{i1} - L_2 W_{i4} + \varsigma_i Y_{i4} - 2 Y_{i5} \\
\Xi_{34} &= \varsigma_i Y_{i2}^T - Y_{i3}^T + Y_{i5}, \ \Xi_{36} = L_2 W_{i6} \\
\Xi_{44} &= -(1 - \varsigma_{Di}) S_{i2} + \varsigma_i S_{i2} - L_2 W_{i5} + \varsigma_i (X_{i4} + Y_{i1}) + 2 Y_{i3} - 2 X_{i4} \\
\Xi_{47} &= L_2 W_{i5}, \ \Xi_{55} = S_{i5} + \varsigma R_{i5} - W_{i4} \\
\Xi_{58} &= D_i^T P_{3i} \tilde{I}_2, \ \Xi_{66} = -W_{i6} \\
\Xi_{77} &= -(1 - \varsigma_{Di}) S_{i4} + \varsigma_i S_{i4} - W_{i5}, \ \Xi_{78} = F_i^T P_{2i} \tilde{I}_2 \\
\Xi_{88} &= Q_{2i} - (1 - \frac{\pi_{ii}}{h_i}) E_i Q_{2i} E_i^T + \tilde{I}_1^2 ((\pi_{ii} P_{i1} + Q_{ii}) + 2 S_{i4} + \delta M_{4} + hR_{4} + R_{4}) - \tilde{I}_1 W_{i1} \\
\Xi_{10,10} &= -(1 - \frac{\pi_{ii}}{h_i}) I_4 Q_{2i} \tilde{I}_1^T - S_{ii} + \varsigma_i h_{i1} S_{ii} - \tilde{I}_1 W_{i3} + h Y_{i4} - 2 V_{i5} \\
\Xi_{10,11} &= h_{i} V_{i2}^T - V_{i3}, \ \Xi_{11,12} = \Xi_{11,13} = L_{i} W_{i3} \\
\Xi_{11,11} &= -(1 - \varsigma_{Di}) S_{i4} + \varsigma_i h_{i1} S_{i4} - \tilde{I}_1 W_{i2} + h_{i} U_{i4} - 2 U_{i5} + 2 V_{i3} + h_{i} V_{i1} \\
\Xi_{11,14} &= L_{i} W_{i2}, \ \Xi_{12,12} = S_{ii} + h R_{6} - W_{i4} \\
\Xi_{13,13} &= -W_{i3}, \ \Xi_{14,14} = -(1 - \varsigma_{Di}) S_{i4} + \varsigma_i h_{i1} S_{i4} - W_{i3} \\
\Xi_{11} &= \tilde{I}_1^T \sum_{j \neq i} \pi_{ij} P_{j1} \tilde{I}_1 + \sum_{j \neq i} \pi_{ij} S_{ij} \tilde{I}_1 Q_{11} \tilde{I}_1^T \\
\Xi_{13} &= \sum_{j \neq i} \pi_{ij} S_{ij} \tilde{I}_1 Q_{11} \tilde{I}_1^T \\
\Xi_{33} &= \sum_{j \neq i} \pi_{ij} S_{ij} \tilde{I}_1 Q_{11} \tilde{I}_1^T + \sum_{j \neq i} \pi_{ij} S_{ij} S_{ij}
\end{align*}\]
\[ \Xi_{14} = \sum_{j \neq i} \pi_{ij} \xi_j S_{2i}, \quad \Xi_{17} = \sum_{j \neq i} \pi_{ij} \xi_j S_{6i} \]

\[ \Xi_{88} = \tilde{I}_2^T \sum_{j \neq i} \pi_{ij} P_{2j} \tilde{I}_2 + \sum_{j \neq i} \pi_{ij} h_j \tilde{E}_2 Q_{2i} \tilde{E}_2^T \]

\[ \Xi_{8,10} = \sum_{j \neq i} \pi_{ij} h_j \tilde{E}_2 Q_{2i} \tilde{I}_4^T \]

\[ \Xi_{10,10} = \sum_{j \neq i} \pi_{ij} h_j S_{3i} + \sum_{j \neq i} \pi_{ij} h_j \tilde{I}_4 Q_{2i} \tilde{I}_4^T \]

\[ \Xi_{11,11} = \sum_{j \neq i} \pi_{ij} h_j S_{4i}, \quad \Xi_{14,14} = \sum_{j \neq i} \pi_{ij} h_j S_{6i} \]

All the other items in matrix $\Xi$ and $\bar{\Xi}$ are 0.

\[ \tilde{I}_1 = [I_n \ 0_{n \times (l-1)n}], \quad \tilde{I}_2 = [I_n \ 0_{n \times (k-1)n}] \]

\[ \tilde{I}_3 = [0_{n \times (l-1)n} \ I_n], \quad \tilde{I}_4 = [0_{n \times (k-1)n} \ I_n] \]

\[ \tilde{E}_1 = \begin{bmatrix} 0 & 0 \ 
I_n & 0 \ 
0 & I_n \ 
... & ... \ 
0 & ... \ 
I_n & 0 \end{bmatrix}_{n \times n} \]

\[ \tilde{E}_2 = \begin{bmatrix} 0 & 0 \ 
I_n & 0 \ 
0 & I_n \ 
... & ... \ 
0 & ... \ 
I_n & 0 \end{bmatrix}_{k \times k} \]

Proof: Construct the following Lyapunov-Krasovskii functional:

$$ V(x_t, y_t, r(t)) = \sum_{m=1}^{8} V_m(x_t, y_t, r(t)) $$

with

$$ V_1(x_t, y_t, r(t)) = x^T(t) P_1(r(t)) x(t) + y^T(t) P_2(r(t)) y(t) $$

$$ V_2(x_t, y_t, r(t)) = \int_{t-\frac{\sigma}{2}}^{t} \gamma_1^T(s) Q_1(r(t)) r_1(s) ds + \int_{t-\delta}^{t} \gamma_2^T(s) Q_2(r(t)) r_2(s) ds $$

where

$$ \gamma_1^T(s) = \begin{bmatrix} x^T(s) & x^T(s - \frac{\sigma}{2}) & ... & x^T(s - \frac{(l-1)\tau}{2}) \end{bmatrix}, $$

$$ \gamma_2^T(s) = \begin{bmatrix} y^T(s) & y^T(s - \frac{\tau}{2}) & ... & y^T(s - \frac{(k-1)\delta}{2}) \end{bmatrix} $$

$$ V_3(x_t, y_t, r(t)) = \int_{t-\sigma}^{t} x^T(s) Q_3(r(t)) x(s) ds + \int_{t-\delta}^{t} y^T(s) Q_4(r(t)) y(s) ds $$

$$ V_4(x_t, y_t, r(t)) = \int_{t-\tau}^{t} x^T(s) S_1(r(t)) x(s) ds + \int_{t-\tau}^{t} x^T(s) S_2(r(t)) x(s) ds + \int_{t-\tau}^{t} y^T(s) S_3(r(t)) y(s) ds + \int_{t-\tau}^{t} y^T(s) S_4(r(t)) y(s) ds $$

$$ V_5(x_t, y_t, r(t)) = \int_{t-\tau}^{t} g^T(x(s)) S_5(r(t)) g(x(s)) ds + \int_{t-\tau}^{t} f^T(y(s)) S_6(r(t)) f(y(s)) ds $$

$$ V_6(x_t, y_t, r(t)) = \int_{t-\tau}^{t} \gamma_1^T(s) M_1 \gamma_1(s) ds d\theta + \int_{t-\tau}^{t} \gamma_2^T(s) M_2 \gamma_2(s) ds d\theta + \int_{t-\tau}^{t} \gamma_3^T(s) M_3 \gamma_3(s) ds d\theta + \int_{t-\tau}^{t} \gamma_4^T(s) M_4 \gamma_4(s) ds d\theta $$

$$ V_7(x_t, y_t, r(t)) = \int_{t-\tau}^{t} x^T(s) (R_1 + R_2) x(s) ds d\theta + \int_{t-\tau}^{t} y^T(s) (R_3 + R_4) y(s) ds d\theta + \int_{t-\tau}^{t} g^T(x(s)) R_5 g(x(s)) ds d\theta + \int_{t-\tau}^{t} f^T(y(s)) R_6 f(y(s)) ds d\theta $$

Then, taking the derivative of $V(x_t, y_t, r(t))$ with respect to $t$ along the system (7) yields

$$ LV_1(x_t, y_t, i) = 2 x^T(t) P_1 \dot{x}(t) + x^T(t) \left( \sum_{j=1}^{N} \pi_{ij} P_{1j} \right) x(t) + 2 y^T(t) P_2 \dot{y}(t) + y^T(t) \left( \sum_{j=1}^{N} \pi_{ij} P_{2j} \right) y(t) $$
\[ LV_2(x_t, y_t, i) = r^T_1(t)Q_1x(t) - r^T_1(t - \frac{S_t}{l})Q_1x(t) + \sum_{j=1}^{N} \frac{\pi_{ij}s_j}{l} r^T_1(t) - \frac{S_s}{l} Q_1x(t) - \frac{S_s}{l} Q_1x(t) + \sum_{j=1}^{N} \frac{\pi_{ij}h_i}{k} r^T_2(t)Q_2x(t) - \frac{h_i}{k} Q_2x(t) - \frac{h_i}{k} Q_2x(t) + \int_{t-h(t)}^{t} y^T(t) \left( \sum_{j=1}^{N} \pi_{ij}s_j \right) y(t)ds \]

\[ (23) \]

\[ LV_3(x_t, y_t, i) = x^T(t)Q_3x(t) - x^T(t - \sigma)Q_3x(t) + \sum_{j=1}^{N} \pi_{ij}s_j x(t)ds + y^T(t)Q_4y(t) - y^T(t - \delta)Q_4y(t) + \int_{t - \delta}^{t} y^T(t) \left( \sum_{j=1}^{N} \pi_{ij}s_j \right) y(t)ds \]

\[ (24) \]

\[ LV_4(x_t, y_t, i) \leq x^T(t)S_1x(t) - x^T(t - \zeta_t)S_1x(t - \zeta_t) + \sum_{j=1}^{N} \pi_{ij}s_j x(t)ds + x^T(t)S_2x(t) - (1 - \zeta) x^T(t)S_2x(t - \zeta) + \sum_{j=1}^{N} \pi_{ij}s_j x(t)ds + y^T(t)S_3y(t) - y^T(t - h_s)S_3y(t - h_s) + \sum_{j=1}^{N} \pi_{ij}h_j y(t)S_3y(t) - \sum_{j=1}^{N} \pi_{ij}h_j y(t)S_3y(t) + \int_{t-h_s}^{t} y^T(t) \left( \sum_{j=1}^{N} \pi_{ij}s_j \right) y(t)ds + y^T(t)S_4y(t) - y^T(t - h_s)S_4y(t - h_s) - \sum_{j=1}^{N} \pi_{ij}h_j y(t)S_3y(t) - \sum_{j=1}^{N} \pi_{ij}h_j y(t)S_3y(t) + \int_{t-h_s}^{t} y^T(t) \left( \sum_{j=1}^{N} \pi_{ij}s_j \right) y(t)ds + y^T(t)S_4y(t) - y^T(t - h_s)S_4y(t - h_s) \]

\[ (25) \]

\[ LV_5(x_t, y_t, i) \leq g^T(x(t))S_5g(x(t)) + f^T(y(t))S_6f(y(t)) - (1 - \zeta) g^T(x(t) - \zeta(t))S_5g(x(t) - \zeta(t)) - (1 - h_s) f^T(y(t) - h_s(t))S_6f(y(t) - h_s(t)) + \sum_{j=1}^{N} \pi_{ij}s_j g^T(x(t))S_5g(x(t)) + \sum_{j=1}^{N} \pi_{ij}s_j f^T(y(t) - h_s(t))S_6f(y(t) - h_s(t)) + \int_{t-h}^{t} g^T(t) \left( \sum_{j=1}^{N} \pi_{ij}s_j \right) g(t)ds + \int_{t-h}^{t} f^T(t) \left( \sum_{j=1}^{N} \pi_{ij}s_j \right) f(t)ds \]

\[ (26) \]

Using Lemma 1 and (15)-(18), one can obtain the following
where

\[ \Sigma_1 = \Xi + \xi X^T R_T \Xi + \hat{h} \Xi^T R_S \Xi \]

According to (19) and Schur complement, we can get \( \Sigma_1 < 0 \), let \( \lambda_1 = \min \lambda_{\text{min}} \{-\Sigma_1\}, i \in S \), so \( \lambda_1 > 0 \). Then, by Dynkin’s formula, we have

\[
E \{ V(x_t, y_t, i) \} - E \{ V(\phi, \varphi, i_0) \} \leq -\lambda_1 E \left\{ \int_0^t (|x(s)|^2 + |y(s)|^2) ds \right\}
\]

and hence

\[
E \left\{ \int_0^t (|x(s)|^2 + |y(s)|^2) ds \right\} \leq \frac{1}{\lambda_1} E \{ V(\phi, \varphi, i_0) \}
\]

Based on Definition 1, the system (7) are stochastically stable and the proof is completed.

**Remark 1** Theorem 1 proposes an improved stochastically stability criterion for Markovian jump BAM neural networks with leakage and discrete delays. The main idea is to divide the delay interval into multiple segments, and the thinner the delay is partitioned, the more obviously the conservatism can be reduced.

Based on Theorem 1, we have the following result for uncertainty Markovian jumping parameters of BAM neural networks with leakage and discrete delays.

**Theorem 2** For any given scalars \( h_1 \geq 0, \delta_1 \geq 0, h_{D1}, \delta_{D1} \) and integers \( l \geq 1, k \geq 1 \), the system (7) with leakage and discrete delays is globally asymptotically stable if there exist two scalars \( \varepsilon_1 > 0, \varepsilon_2 > 0 \), symmetric positive definite matrices \( P_{ij}, Q_{ij}, M_j, (j = 1, 2, 3, 4), S_j, (j = 1, 2, \ldots, 6), R_j, (j = 1, 2, \ldots, 8) \), positive diagonal matrices \( W_{ij}, (j = 1, 2, \ldots, 6) \), and any matrices

\[
X_i = \begin{bmatrix} X_{i1} & X_{i2} & X_{i3} \\ * & X_{i4} & X_{i5} \\ * & * & X_{i6} \end{bmatrix},
Y_i = \begin{bmatrix} Y_{i1} & Y_{i2} & Y_{i3} \\ * & Y_{i4} & Y_{i5} \\ * & * & Y_{i6} \end{bmatrix},
U_i = \begin{bmatrix} U_{i1} & U_{i2} & U_{i3} \\ * & U_{i4} & U_{i5} \\ * & * & U_{i6} \end{bmatrix}
\]

with appropriate dimensions, for any \( i = 1, 2, \ldots, N \), such that the following LMIs hold:

\[
\sum_{j=1}^{N} \pi_{ij} Q_{kj} < M_k, \quad k = 1, 2, 3, 4
\]

\[
\sum_{j=1}^{N} \pi_{ij} S_{kj} < R_k, \quad k = 1, 2, \ldots, 6
\]
\[
\begin{bmatrix}
V_{1i} & V_{2i} & V_{3i} \\
\ast & V_{4i} & V_{5i} \\
\ast & \ast & R_{8i}
\end{bmatrix} \geq 0
\]  
\tag{41}

\[
\begin{bmatrix}
\Xi + \Xi & \Xi T R_7 & \Xi T R_8 \\
\ast & -\Xi T R_7 & 0 \\
\ast & \ast & -\Xi T R_8
\end{bmatrix} + \begin{bmatrix}
\Xi T & \Xi T & \Xi T \\
\ast & 0 & \ast \\
\ast & \ast & 0
\end{bmatrix} + \begin{bmatrix}
\Xi T & \Xi T & \Xi T \\
\ast & 0 & \ast \\
\ast & \ast & 0
\end{bmatrix} < 0
\]  
\tag{42}

where
\[
N_{11} = \begin{bmatrix} 0 & -E_{i1} & 0_{n \times 9n} & E_{ci} & 0 & E_{ci} \end{bmatrix}
\]
\[
N_1 = \begin{bmatrix} N_{11} & 0 \end{bmatrix}
\]
\[
N_{22} = \begin{bmatrix} G_1^T P_1 \tilde{I}_1 & 0_{n \times 13n} \end{bmatrix}^T
\]
\[
N_2 = \begin{bmatrix} N_{22} & \frac{1}{2} G_1^T R_7 & 0 \end{bmatrix}^T
\]
\[
Z_{11} = \begin{bmatrix} 0_{n \times 4n} & E_{di} & 0 & E_{fi} & 0 & -E_{bi} & 0_{n \times 5n} \end{bmatrix}
\]
\[
Z_1 = \begin{bmatrix} Z_{11} & 0 \end{bmatrix}
\]
\[
Z_{22} = \begin{bmatrix} 0_{n \times 8n} & G_1^T P_2 \tilde{I}_2 & 0_{n \times 6n} \end{bmatrix}^T
\]
\[
Z_2 = \begin{bmatrix} Z_{22} & \frac{1}{2} G_1^T R_8 \end{bmatrix}^T
\]

Proof: Replacing $A_1, B_1, C_1, D_1, E_1, F_1$ in (19) with
\[
A_i + G_i F_i(t) E_{ai}, B_i + G_i F_i(t) E_{bi}, C_i + G_i F_i(t) E_{ci}, D_i + G_i F_i(t) E_{di}, F_i + G_i F_i(t) E_{fi}, \text{respectively.}(19) \text{ is equivalent to the following condition:}
\]
\[
\Xi + \Xi T R_7 \tilde{I} R_8 + \Xi T R_7 R_8 + \Xi T R_7 \Xi
\]
\[
\ast \ast -\Xi T R_7 0 \\
\ast \ast -\Xi T R_8
\]
\tag{43}

According to Lemma 2, (43) is true if there exist two scalars $\varepsilon_1, \varepsilon_2 > 0$ such that the following inequality holds:
\[
\Xi + \Xi T R_7 \tilde{I} R_8 + \varepsilon_1 \Xi T R_7 \Xi
\]
\[
\ast \ast -\Xi T R_7 0 \\
\ast \ast -\Xi T R_8
\]  
\tag{44}

Using the Schur complement shows that (44) is equivalent to (42). This completes the proof.

Remark 2 In this paper, Theorem 1 and Theorem 2 require the upper bound of the derivative of time-varying $h_{D_i}, \xi_{D_i}$ known. However, in practice, $h_{D_i}, \xi_{D_i}$ are unknown. Considering this situation, we can set $S_{ji} = 0, j = 1, 2, \ldots, 6$ in Theorem 1 and Theorem 2.

### Table I

<table>
<thead>
<tr>
<th>Method</th>
<th>$h_D = 0.1$</th>
<th>$h_D = 0.3$</th>
<th>$h_D = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l = 1, k = 1$</td>
<td>0.692</td>
<td>0.541</td>
<td>0.437</td>
</tr>
<tr>
<td>$l = 1, k = 2$</td>
<td>1.573</td>
<td>1.268</td>
<td>1.025</td>
</tr>
<tr>
<td>$l = 2, k = 3$</td>
<td>1.917</td>
<td>1.901</td>
<td>1.873</td>
</tr>
<tr>
<td>$l = 3, k = 4$</td>
<td>2.589</td>
<td>2.448</td>
<td>2.098</td>
</tr>
</tbody>
</table>

### Table II

<table>
<thead>
<tr>
<th>Method</th>
<th>$\xi_D = 0.1$</th>
<th>$\xi_D = 0.3$</th>
<th>$\xi_D = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l = 1, k = 1$</td>
<td>0.753</td>
<td>0.675</td>
<td>0.542</td>
</tr>
<tr>
<td>$l = 1, k = 2$</td>
<td>1.178</td>
<td>1.025</td>
<td>0.978</td>
</tr>
<tr>
<td>$l = 2, k = 3$</td>
<td>1.769</td>
<td>1.561</td>
<td>1.252</td>
</tr>
<tr>
<td>$l = 3, k = 4$</td>
<td>2.364</td>
<td>2.237</td>
<td>2.034</td>
</tr>
</tbody>
</table>

### IV. Example

In this section, we provide one numerical example to demonstrate the effectiveness and less conservatism of our delay-dependent stability criteria.

**Example 1** Consider delayed BAM neural networks with uncertainty Markovian jumping parameters as follows:
\[
\begin{align*}
\dot{x}(t) &= -A_1 x(t - \sigma) + C_1 f(y(t)) + E_1 f(y(t - h_1(t))) \\
\dot{y}(t) &= -B_1 y(t - \delta) + D_1 g(x(t)) + F_1 g(x(t - h_2(t)))
\end{align*}
\]

where
\[
A_1 = \begin{bmatrix} 1.8 & 0 \\ 0 & 2.2 \end{bmatrix}, B_1 = \begin{bmatrix} 2.3 & 0 \\ 0 & 1.6 \end{bmatrix}, C_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, D_1 = \begin{bmatrix} 0.4 & -0.3 \\ -0.8 & 0.1 \end{bmatrix},
\]
\[
E_1 = \begin{bmatrix} 0.3 & 0.6 \\ -0.5 & -0.9 \end{bmatrix}, F_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.4 \end{bmatrix}, F_2 = \begin{bmatrix} -0.4 & 0.1 \\ 0.1 & -0.7 \end{bmatrix},
\]
\[
\pi = \begin{bmatrix} -7 & 7 \\ 6 & -6 \end{bmatrix}
\]

In this example, we assume condition $\sigma = \delta = 0.1$. In Table I, we consider the case of $h_1 = h_2 = 0.1$, the upper bound of $\xi$ with different $l, k, \xi_D$, unknown. In Table II, we consider the other case of $\xi_1 = \xi_2 = 0.3$, the upper bound of $\tilde{h}$ with different $l, k, \xi_D$. According to this two Tables, we can see this example shows that the stability condition gives much less conservative results in this paper.

### V. Conclusion

In this present paper, we have investigated the problem of stability for uncertainty Markovian jumping parameters of BAM neural networks with leakage and discrete delays. Two sufficient conditions have been presented. The obtained criteria are less conservative because free-weighting matrices method and a convex optimization approach are considered. Finally, one example has been given to illustrate the effectiveness of the proposed method.
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REFERENCES


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