Common Fixed Point Theorems for Co-cyclic Weak Contractions in Compact Metric

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Abstract—In this paper, we prove some common fixed point theorems for co-cyclic weak contractions in compact metric spaces.

Keywords—Cyclic weak contraction, Co-cyclic weak contraction, Co-cyclic representation, Common fixed point.

I. INTRODUCTION


Beg et. al. [4] and Babu et. al. [3] proved common fixed point theorems for a pair of weakly contractive map in complete metric space.

In 2003, Kirk et. al. [9] introduced the notion of Cyclic contraction and established some related fixed point theorems for mappings satisfying such contraction conditions. Suggested by the consideration in [9], Rus [11] introduced the following concept of cyclic representation and proved some fixed point theorems.

Definition 1: [11] Let X be a nonempty set, m a positive integer and T : X → X a selfmap. X = \bigcup_{i=1}^{m} A_i is said to be a cyclic representation of X with respect to the map T if the following conditions hold:
1) A_i, i = 1, 2, ..., m are nonempty subsets of X;
2) T(A_1) ⊆ A_2, ..., T(A_{m-1}) ⊆ A_m, T(A_m) ⊆ A_1.

In [8], Karapinar proves a fixed point theorem for a mapping T defined on a complete metric space X when X has a cyclic representation with respect to T.

Example 1: [5] Let X = [0, 2], A_1 = [0, 1], A_2 = [1/2, 2] and A_3 = [1, 2]. Now, we define a selfmap T on X by

\[ T(x) = \begin{cases} 
  x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}] \\
  1 & \text{if } x \in (\frac{1}{2}, 2] \\
  x - 1 & \text{if } x \in (\frac{1}{2}, 2]. 
\end{cases} \]

Then we observe that T(A_1) = [1/2, 1] ⊆ [1/2, 2] = A_2, T(A_2) ⊆ [1, 2] = A_3 and T(A_3) = (1/2, 1] ⊆ [0, 1] = A_0.

Therefore, X = \bigcup_{i=1}^{3} A_i is a cyclic representation of X with respect to T.

Throughout this paper, we denote R_+ = [0, \infty) and \bar{\mathbb{J}} = \{ \varphi \mid \varphi : R_+ \rightarrow R_+ \text{ is nondecreasing, } \varphi(0) = 0, \varphi(t) > 0 \text{ for } t > 0 \}.

Recently, Harjani et.al. [6] established the following fixed point theorem for a continuous selfmap.

Theorem 1: Let (X, d) be a compact metric space and T : X → X a continuous operator. Suppose that m is a positive integer, A_1, A_2, ..., A_m nonempty subsets of X, X = \bigcup_{i=1}^{m} A_i satisfying
1) X = \bigcup_{i=1}^{m} A_i is a cyclic representation of X with respect to T;
2) d(Tx, Ty) ≤ d(x, y) − \varphi(d(x, y)) for any x ∈ A_i and y ∈ A_{i+1}, where \varphi ∈ \bar{\mathbb{J}}.

Then T has a unique fixed point.

Note that to guarantee the existence and uniqueness of common fixed points of a pair of maps, we need an additional condition, called weak compatibility, which is defined as follows.

Definition 2: [7] Let X be a nonempty set. Two selfmaps S, T : X → X are said to be weakly compatible if they commute at their coincidence points, i.e., if x ∈ X such that Sv = Tx, then STx = TSx.

The purpose of this paper is to establish a common fixed point theorem for a co-cyclic weak contraction defined in compact metric spaces. Our result extends the result of Harjani et. al. [6] to a co-cyclic weak contraction.

II. PRELIMINARIES

Definition 3: [5] Let X be a nonempty set, m a positive integer and T, f : X → X be two selfmaps. X = \bigcup_{i=1}^{m} A_i is said to be a co-cyclic representation of X between f and T if the following conditions are satisfied:
1) A_i, i = 1, 2, ..., m are nonempty subsets of X;
2) T(A_1) ⊆ f(A_2), ..., T(A_{m-1}) ⊆ f(A_m), and T(A_m) ⊆ f(A_1).

Example 2: Let X = [0, 1], and A_1 = [0, 1/2] and A_2 = [1/2, 1]. We define a selfmap T and f on X by

\[ T(x) = \begin{cases} 
  x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}] \\
  1 - x & \text{if } x \in (\frac{1}{2}, 1]. 
\end{cases} \]

and

\[ f(x) = \begin{cases} 
  x & \text{if } x \in [0, \frac{1}{2}] \\
  2x - 1 & \text{if } x \in (\frac{1}{2}, 1]. 
\end{cases} \]

Then we observe that T(A_1) = [1/2, 1] ⊆ [0, 1] = f(A_2), T(A_2) = [1/2, 1] = f(A_1). Therefore, X = \bigcup_{i=1}^{2} A_i is a co-cyclic representation of X between f and T.

We now introduce the following definitions.

Definition 4: Let (X, d) be a metric space, m a positive integer, A_1, A_2, ..., A_m a closed nonempty subsets of X, and X = \bigcup_{i=1}^{m} A_i. An operator T : X → X is said to be co-cyclic weak contraction if there is an operator f : X → X such that

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1) \( X = \bigcup_{i=1}^{m} A_i \) is a cyclic representation of \( X \) between \( f \) and \( T \);
2) \( d(Tx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy)) \) for any \( x \in A_i \) and \( y \in A_{i+1} \), where \( A_{m+1} = A_1 \) and \( \varphi \in \mathfrak{I} \).

The purpose of this paper is to prove the following theorem.

### III. MAIN RESULTS

**Theorem 2:** Let \( (X, d) \) be a compact metric space and \( f, T : X \to X \) be two continuous operators. Suppose that \( m \) is a positive integer, \( A_1, A_2, \cdots, A_m \) are nonempty subsets of \( X \), and \( X = \bigcup_{i=1}^{m} A_i \), satisfying

1) \( X = \bigcup_{i=1}^{m} A_i \) is a co-cyclic representation of \( X \) between \( f \) and \( T \);
2) \( d(Tx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy)) \) for any \( x \in A_i \) and \( y \in A_{i+1} \), where \( A_{m+1} = A_1 \) and \( \varphi \in \mathfrak{I} \).

If the pair of operators \( (f, T) \) are weakly compatible on \( X \), then \( f \) and \( T \) have a unique common fixed point in \( X \).

**Proof:** Let \( x_0 \in X \). Since \( T(A_i) \subset f(A_{i+1}) \) for each \( i = 1, 2, \cdots, m-1 \) and \( T(A_m) \subset f(A_1) \), there exists \( x_1 \in X \) such that \( Tx_0 = fx_1 \). On continuing the process, inductively we get a sequence \( \{x_n\} \subseteq X \) such that \( Tx_n = fx_{n+1} \) for each \( n = 0, 1, 2, \cdots \).

If there exists \( n_0 \in \mathbb{N} \) with \( Tx_{n_0+1} = Tx_{n_0} = fx_{n_0+1} \) and, thus, \( f \) and \( T \) have coincidence point \( x_{n_0+1} \).

Suppose that \( x_{n+1} \neq x_n \) for all \( n = 0, 1, 2, \cdots \). We now show that the sequence \( \{d(fx_n, fx_{n+1})\} \) is a nonincreasing sequence. By (1) of Theorem 3.1, \( \{d(fx_n, fx_{n+1})\} \) is a nonincreasing sequence of nonnegative reals and hence converges to a limit \( l \geq 0 \). Letting \( n \to \infty \) in (1), we obtain

\[
0 < \varphi(l) \leq \lim_{n \to \infty} \varphi(d(fx_n, fx_{n+1}))
\]

which is a contradiction to (3). Therefore, \( l = 0 \).

Hence,

\[
\lim_{n \to \infty} d(fx_n, fx_{n+1}) = 0
\]

Since \( Tx_n = fx_{n+1} \) for each \( n = 1, 2, \cdots \), from (4) it follows that

\[
\inf\{d(fx_n, Tx) : x \in X\} = 0.
\]

Now show the sequence \( \{d(fx_n, fx_{n+1})\} \) is a nonincreasing sequence and \( f, T \) are weakly compatible, we get

\[
Tz = Tfz = fz.
\]

We claim that \( z = Tz \). Suppose \( z \neq Tz \). Then,

\[
d(z, Tz) = d(Tu, Tu) \\
\leq d(fu, fx) - \varphi(d(fu, fx)) \\
\leq d(z, Tu) - \varphi(d(z, Tu)),
\]

which shows that

\[
\varphi(d(z, Tz)) \leq 0.
\]

But

\[
\varphi(d(z, Tz)) \geq 0.
\]

Hence,

\[
\varphi(d(z, Tz)) = 0
\]

and since \( \varphi \in \mathfrak{I} \), we have

\[
d(z, Tz) = 0.
\]

Hence,

\[
Tz = z.
\]

Hence, by (6), we obtain

\[
fz = Tz = z.
\]

For the uniqueness part, suppose that \( z \) and \( w \) are common fixed points of \( f \) and \( T \). Since \( X = \bigcup_{i=1}^{m} A_i \) is co-cyclic representation of \( X \) between \( f \) and \( T \), we have \( z, w \in \bigcap_{i=1}^{m} A_i \).

\[
d(z, w) = d(Tz, Tw) \leq d(fz, fw) - \varphi(d(fz, fw)) \\
\leq d(z, w) - \varphi(d(z, w))
\]

Therefore,

\[
\varphi(d(z, w)) = 0.
\]

Since \( \varphi \in \mathfrak{I} \), \( d(z, w) = 0 \) and hence, \( z = w \).
Since the identity map $I_X$ defined on $X$ is weakly compatible with any selfmap $T$ defined on $X$, if we choose $f = I_X$, the identity map on $X$, we obtain the following result:

**Corollary 1:** Let $(X, d)$ be a compact metric space and $T : X \to X$ be a continuous operator. Suppose that $m$ is a positive integer, $A_1, A_2, \ldots, A_m$ are nonempty subsets of $X$, and $X = \bigcup_{i=1}^{m} A_i$, satisfying

1) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of $X$ with respect to $T$;
2) $d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$ for any $x \in A_i$ and $y \in A_{i+1}$, where $A_{m+1} = A_1$ and $\varphi \in \mathcal{J}$.

Then $T$ has a unique fixed point in $X$.

**Proof.** Follows from Theorem 1 by choosing $f = I_X$.

**Remark** We observe that Theorem 1 extends Theorem 2.1 of [6] to co-cyclic weak contraction.

**Definition 5:** [1] Let $(X, d)$ be a metric space, and $\Xi$ be a set of selfmappings of $X$. The common fixed points of the set $\Xi$ is said to be well-posed if:

1) $\Xi$ has a unique common fixed point in $X$ (That is, $z$ is the unique point in $X$ such that $Tz = z$ for all $T \in \Xi$);
2) For every sequence $\{z_n\}$ in $X$ such that

$$\lim_{n \to \infty} d(z_n, Tz_{n+1}) = 0, \forall T \in \Xi,$$

we have

$$\lim_{n \to \infty} d(z_n, z) = 0.$$

Our second result is concerned with the well-posedness of the common fixed point problem for two mappings $f$ and $T$ satisfying the inequality (2) of Theorem 1.

**Theorem 3:** Under the assumptions of Theorem 1, the common fixed point problem for $f$ and $T$ is well-posed; that is, if there is a sequence $\{z_n\}$ in $X$ with $d(z_n, Tz_n) \to 0$ and $d(z_n, fz_n) \to 0$ as $n \to \infty$, then $z_n \to z$ as $n \to \infty$, where $z$ is the unique common fixed point of $f$ and $T$ (whose existence is guaranteed by Theorem 1).

**Proof.** By Theorem 1, $f$ and $T$ have a unique common fixed point $z$. As $z$ is common fixed point of $f$ and $T$, by (2) of Theorem 1, $z \in \bigcap_{i=1}^{m} A_i$. Let $\{z_n\}$ be a sequence in $X$ such that $d(z_n, Tz_n) \to 0$ and $d(z_n, fz_n) \to 0$ as $n \to \infty$. Now consider

$$d(z, Tz_n) \leq d(z, fz_n) - \varphi(d(z, fz_n))$$

(7)

$$\leq d(z, z_n) + d(z_n, fz_n) - \varphi(d(z, fz_n))$$

(8)

Also, from the triangle inequality, (2) of Theorem 1, Equation (8) and the fact that $z \in \bigcap_{i=1}^{m} A_i$, we have

$$d(z, z_n) \leq d(z, Tz_n) + d(Tz_n, z_n)$$

(9)

$$\leq d(z, z_n) + d(z_n, fz_n)$$

$$- \varphi(d(z, fz_n)) + d(Tz_n, z_n)$$

which implies

$$\varphi(d(z, fz_n)) \leq d(z_n, fz_n) + d(Tz_n, z_n) \to 0$$

as $n \to \infty$. Hence,

$$\lim_{n \to \infty} \varphi(d(z_n, z)) = 0.$$  

(10)

We now claim that $\lim_{n \to \infty} d(fz_n, z) = 0$. Suppose not. Then there exists $\varepsilon \geq 0$ such that for any $n \in \mathbb{N}$ we can find $k_n \geq n$ with $d(fz_{k_n}, z) \geq \varepsilon$. Since $\varphi \in \mathcal{J}$ is nondecreasing and $\varphi(t) > 0$ for $t \in (0, \infty)$, we have

$$0 < \varphi(\varepsilon) \leq \varphi(d(fz_{k_n}, z)).$$

(11)

Letting $n \to \infty$ in (11)

$$0 < \varphi(\varepsilon) \leq \lim_{n \to \infty} \varphi(d(fz_{k_n}, z)).$$

which contradicts (10). Therefore,

$$\lim_{n \to \infty} d(fz_n, z) = 0$$

and hence letting $n \to \infty$ in (7), we obtain

$$\lim_{n \to \infty} d(z, Tz_n) = 0.$$  

(12)

Consequently, letting $n \to \infty$ in (9), using (12) we obtain

$$\lim_{n \to \infty} d(z_n, z) = 0$$

Hence the common fixed point problem of $f$ and $T$ is well-posed.

**REFERENCES**


