Another Structure of Weakly Left C-wrpp Semigroups
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Abstract—It is known that a left C-wrpp semigroup can be described as curler structure of a left band and a C-wrpp semigroup. In this paper, we introduce the class of weakly left C-wrpp semigroups which includes the class of weakly left C-rpp semigroups as a subclass. We shall particularly show that the spined product of a left C-wrpp semigroup and a right normal band is a weakly left C-wrpp semigroup. Some equivalent characterizations of weakly left C-wrpp semigroups are obtained. Our results extend that of left C-wrpp semigroups.

Keywords—Left C-wrpp semigroup, left quasi normal regular band, weakly left C-wrpp semigroup.

I. INTRODUCTION

THROUGHOUT this paper, we adopt the notation and terminologies given by Howe[1] and Du[2].

Tang[3] considered a Green-like right congruence relation \( \mathcal{L}^* \) on a semigroup \( S \) : for \( a, b \in S, a\mathcal{L}^*b \) if and only if \( ax=ay \Leftrightarrow bx=by \) for all \( x, y \in S^1 \). Moreover, Tang pointed out in [3] that a semigroup \( S \) is a wrpp semigroup if and only if each \( \mathcal{L}^* \)-class of \( S \) contains at least one idempotent.

Recall that a wrpp semigroup \( S \) is a C-wrpp semigroup if the idempotents of \( S \) are central. It is well known that a semigroup \( S \) is a C-wrpp semigroup if and only if \( S \) is a strong semilattice of left-R cancellative monoids (see[4-9]).

Because a Clifford semigroup can be expressed as a strong semilattice of groups and a C-rpp semigroup can be expressed as a strong semilattice of left cancellative monoids (see[4-9]), we see immediately that the concept of C-wrpp semigroups is a common generalization of Clifford semigroups and C-rpp semigroups.

For wrpp semigroups, Du-Shum [2] first introduced the concept of left C-wrpp semigroups, that is, a left C-wrpp semigroup whose satisfy the following conditions: (i) for all \( e \in E(L^*_a) \), \( a = ae \), where \( E(L^*_a) \) is the set of idempotents in \( L^*_a \); (ii) for all \( a \in S \), there exists a unique idempotent \( a^* \) satisfying \( a^*a = a \) and \( a = a^*a \); (iii) for all \( a \in S, aS \subseteq L^* \), where \( L^* \) is the smallest left \(*\)-ideal of \( S \) generated by \( a \). For such semigroups, Du-Shum[2] gave a method of construction.

Zhang[10] showed that the spined product of a left C-wrpp semigroup and a right normal band which is a weakly left C-wrpp semigroup by virtue of left C-full Ehrenmann cybergroups. In this paper, we first define the concept of weakly left C-wrpp semigroups. A equivalent descriptions of weakly left C-wrpp semigroups is therefore obtained and our results generalize that of Cao on weakly left C-rpp in[5].

In view of the theorems given in this paper, one can easily observe that the results of weakly left C-wrpp semigroups are a common generalizations of weakly left C-semigroups and left C-wrpp semigroups in range of wrpp semigroups.

II. PRELIMINARIES

We first recall some known results used in the sequel. To start with, we introduce the concept of simi-spined product.

Let \( T = \bigcup_{\alpha \in Y} T_\alpha \) and \( I = \bigcup_{\alpha \in Y} I_\alpha \) be the semilattice decomposition of the semigroups \( T \) and \( I \) with respect to semilattice \( Y \) respectively. For all \( \alpha \in Y \), we denote the direct product \( T_\alpha \times T_\alpha \) by \( S_\alpha \). Let \( S = \bigcup_{\alpha \in Y} S_\alpha \). We define the mapping \( \eta \) by the following rules:

\[ \eta : S \to T(I), (i, a) \mapsto \eta(i, a), \eta(i, a) : I \to I, j \mapsto (i, a)^*j, \]

where \( T(I) \) is a left transformation semigroup on \( I \). Suppose that the mapping \( \eta \) satisfies the following conditions:

\[ (Q1) \text{If } (i, a) \in S_{\alpha}, j \in I_{\beta}, \text{ then } (i, a)^*j \in I_{\beta}; \]

\[ (Q2) \text{If } (i, a) \in S_{\alpha}, (j, b) \in S_{\beta} \text{ with } \alpha \leq \beta, \text{ then } (i, a)^*j = ij, \text{ where } ij \text{ is the semigroup product in the semigroup } I = \bigcup_{\alpha \in Y} I_{\alpha}; \]

\[ (Q3) \text{If } (i, a) \in S_{\alpha}, (j, b) \in S_{\beta}, \text{ then } \eta(i, a)\eta(j, b) = \eta(i, a)^*j, ab, \text{ where } ab \text{ is the semigroup product in the semigroup } T = \bigcup_{\alpha \in Y} T_{\alpha}. \]

Then we define a multiplication " \( \circ \) " on \( S = \bigcup_{\alpha \in Y} S_{\alpha} \) by \( (i, a) \circ (j, b) = (i, (a)^*j, ab) \). By a straightforward verification, we can prove that the multiplication " \( \circ \) " satisfies the associative law and hence \( (S, \circ) \) becomes a semigroup, denoted by \( S = I \times \eta T \). We call this semigroup the semi-spined product of \( I \) and \( T \) with respect to the structure mapping \( \eta \).

Lemma 1[2] Let \( I \) be a left regular band which is expressed as a semilattice of left zero bands \( I_\alpha \) (that is, \( I = \bigcup_{\alpha \in Y} I_\alpha \)) and let \( T = \bigcup_{\alpha \in Y} T_\alpha \) be a C-wrpp semigroup (that is, \( T \) is a strong semilattice of left-R cancellative monoids \( Y; T_\alpha, \phi_{\alpha, \beta} \))(see[3]). If the structure mapping \( \eta \) satisfies the following condition:

\[ (Q) : \ker \eta(i, a) = \ker \eta(j, b) \]

for every \( (i, a), (j, b) \in S_\alpha \). Then \( S \) is a left C-wrpp semigroup. Conversely, every left C-wrpp semigroup \( S \) can be constructed in terms of above method.

Lemma 2[5] A semigroup \( S \) is a weakly left C-semigroup, that is, \( S \) is a regular semigroup and

\[ (\forall e \in E(S))\eta^*_e : S \to eS, x \mapsto ex \]
is a homomorphism if and only if $S$ is a completely regular and $E(S)$ is a left quasi-normal band.  

**Lemma 3** [2] If $S$ is a left C-wrpp, then $\text{Reg} S$ is a left C-semigroup.

**Lemma 4** [7] A band $B$ is a left normal band (that is, a band satisfies identity $efg = efg$) if and only if Green relation $\mathcal{L}$ and $\mathcal{R}$ are congruence on $B$ and $B/R$ is a right normal band.

**Definition 1** A monotoid $T$ is a left-$\mathcal{R}$ cancellative monoid if for $a,b,c \in T$, $(ab,ac) \in \mathcal{R}$ implies $(b,c) \in \mathcal{R}$. We call the direct product of a left-$\mathcal{R}$ cancellative monoid $T$ and a rectangular band $I$ a left cancellative plank because the direct product looks like a two-dimensional plank. We denote the left-$\mathcal{R}$ cancellative plank by $I \times T$.

**Lemma 5** [2] Let $I = \cup_{\alpha \in Y} I_{\alpha}$ be a semilattice of left zero bands, and $T = \{ Y : I_{\alpha} , \phi_{\alpha,\beta} \}$ a strong semilattice of left-$\mathcal{R}$ cancellative monoids $T_{\alpha}$. Then $(i,a) \mathcal{R} (j,b)$ if and only if $a \mathcal{R} b$ and $i = j$ for any $(i,a), (j,b) \in S = \cup_{\alpha \in Y}(I_{\alpha} \times T_{\alpha})$.

### III. THE WEAKLY LEFT C-WRPP SEMIGROUPS

In this section, the concept of weakly left C-wrpp semigroups is introduced. We shall give equivalent characterization for the structure of weakly left C-wrpp semigroups. First, we introduce the concept of weakly left C-wrpp semigroups.

**Definition 2** A semigroup $S$ is called a weakly left C-wrpp semigroup, if $S$ is isomorphic to a semilattice of left-$\mathcal{R}$ cancellative planks, and

$$\forall \in E(S) \eta_{\alpha} : S \to eS, x \mapsto ex$$

is a homomorphism. We now characterize the weakly left C-wrpp semigroups.  

**Theorem 1** Let $S$ be a semigroup. Then the following conditions are equivalent:

1. $S$ is a weakly left C-wrpp semigroup;
2. $(S)$ is a semilattice of left-$\mathcal{R}$ cancellative planks, and $\text{Reg} S$ is a weakly left C-semigroup;
3. $(S)$ is a semilattice of left-$\mathcal{R}$ cancellative planks, and $E(S)$ is a left quasi-normal band;
4. $(S)$ is a spined product of left C-wrpp semigroup and a right normal band.

**Proof.** (1) $\Rightarrow$ (2). We only need show that $\text{Reg} S$ is a weakly left C-semigroup. Let $a,b \in \text{Reg} S$. Then there exists $x,y \in S$ such that $a = axa, x = xax, b = bby$. So $e = xa \in E(S)$. According to (1), we know that $\eta_{\alpha}^{\prime}$ is a semigroup homomorphism from $S$ to $eS$. Thus

$$ab = axbyb = a \eta_{\alpha}^{\prime}[(by)\eta_{\alpha}^{\prime}b] = a \eta_{\alpha}^{\prime}(by)\eta_{\alpha}^{\prime}(b) \equiv abxyab \equiv (ab)(xy)(ab)$$

So $ab \in \text{Reg} S$. Therefore, $\text{Reg} S$ is a subsemigroup of $S$. Again $E(\text{Reg} S) = E(S)$, according to Lemma 3, we obtain $\text{Reg} S$ is a weakly left C-semigroup.

(2) $\Rightarrow$ (3). Clearly, we omit it.

(3) $\Rightarrow$ (4). Let $S = \cup_{\alpha \in Y}(I_{\alpha} \times T_{\alpha} \times \Lambda_{\alpha})$ is a semilattice decomposition $Y$ of left-$\mathcal{R}$ cancellative planks, and $E(S)$ is a left quasi-normal band, and put $S_{l} = \cup_{\alpha \in Y}(I_{\alpha} \times T_{\alpha} \times \Lambda_{\alpha})$, where $I_{\alpha}, T_{\alpha}$ and $\Lambda_{\alpha}$ are left zero band, a left-$\mathcal{R}$ cancellative monoid and a right zero band, respectively. Next, we verify that $S_{l} = \cup_{\alpha \in Y}(I_{\alpha} \times T_{\alpha})$ is a left C-wrpp semigroup, and $\Lambda = \cup_{\alpha \in Y} \Lambda_{\alpha}$ is a right normal band.

Step 1 Let $T = \cup_{\alpha \in Y} T_{\alpha}$, we shall show that $T$ is a C-wrpp semigroup, and $\Lambda = \cup_{\alpha \in Y} \Lambda_{\alpha}$ is a right normal band. For this purpose, we only need to show that $T$ is a strong semilattice of left-$\mathcal{R}$ cancellative monoids $T_{\alpha}$, and a strong semilattice of right zero bands $\Lambda_{\alpha}$, respectively.

Identify $T_{\alpha}$ denoted by $I_{\alpha}$, obviously, we have $E(S) = \{(i_1,\alpha,\lambda) | (i_1) \in I_{\alpha} \times \lambda, \alpha \in \Lambda \}$, and

$$\begin{align*}
(i_1,\alpha,\lambda)E(j_1,\beta,\mu) & \Leftrightarrow \alpha = \beta, \lambda = \mu, \\
(i_1,\alpha,\lambda)E(j_1,\beta,\mu) & \Leftrightarrow \alpha = \beta, i = j
\end{align*}$$

(1) where $\mathcal{E}$ and $\mathcal{E}^{R}$ are Green’s relations on semigroup $E(S)$.

For all $\alpha \geq \beta$, let $a = (i,g,\lambda) \in S_{\alpha}$, if $(j,\mu) \in I_{\beta} \times \Lambda_{\beta}$, then there exists $(j_1,h_1,\mu_1) \in S_{\beta}$ such that $(j,1,\mu)a = (j_1,h_1,\mu_1)$. Since $(j_1,h_1,\mu_1) = (j_1,1,\mu_1)(j_1,\mu_1)a = (j_1,1,\mu_1)$, we obtain $j_1 = j$. On the other hand, for all $j \in I_{\beta}$, we have $(j,1,\mu)a = (j_1,1,\mu_1) = (j_1,\mu_1)(j_1,\mu_1)a = (j_1,\mu_1,\mu_1)$, and $j_1, \mu_1$ do not depend on the choice of $j$ in $I_{\beta}$. Let $\Lambda_{1} = \mu(i,g,\lambda)\chi_{\beta,\alpha}, \mu_{1} = \mu(i,g,\mu)\psi_{\alpha,\beta}$. Then we have

$$\begin{align*}
(j,1,\mu)(i,g,\lambda) = (j,\mu(i,g,\lambda)\chi_{\beta,\alpha},\mu(i,g,\mu)\psi_{\alpha,\beta}.
\end{align*}$$

(3)

Similarly, we show that there exists $\phi_{\beta,\alpha}(i,g,j) \in I_{\beta}, \varphi_{\beta,\alpha}(i,g,j) \in T_{\beta}$ such that

$$\begin{align*}
(i,g,\lambda)(j,1,\mu) = (\phi_{\beta,\alpha}(i,g,j),\varphi_{\beta,\alpha}(i,g,j)).
\end{align*}$$

(4)

where $\Lambda_{\beta}, \alpha \geq \beta$. Similarly, by $\mathcal{E}^{R}$ is a congruence on $E(S)$, we follow that $\mu(i,g,\lambda)\chi_{\beta,\alpha}, \mu(i,g,\mu)\psi_{\alpha,\beta}$ do not depend on the choice of $i$ in $I_{\alpha}$, let

$$\begin{align*}
\phi_{\beta,\alpha}(i,g,j) = \phi_{\beta,\alpha}(i,g,j), \varphi_{\beta,\alpha}(i,g,j) = \varphi_{\beta,\alpha}(i,g,j).
\end{align*}$$

(5)
homomorphism of from \( T_\alpha \) to \( T_\beta \) and from \( \Lambda_\alpha \) to \( \Lambda_\beta \), respectively, where \( \alpha \geq \gamma \). Similarly, it follows that \( \sigma_{\alpha,\beta} \) is also a semigroup homomorphism, by (9), we have

\[
1 \circ \sigma_{\alpha,\beta} = 1 \circ \beta (\alpha \geq \beta).
\]  
(15)

(ii) If \( \beta = \alpha \), let \( \gamma = \alpha, h = 1_{\alpha}, \mu = \lambda \). In view of (14) and (15), it follows that \( g = \sigma_{\alpha,\alpha} : \chi = \lambda_\alpha \alpha \) for any \( g \in T_\alpha \), \( \lambda \in \Lambda_\alpha \). So \( \sigma_{\alpha,\alpha} \) and \( \theta_{\alpha,\beta} \) are identical mapping on \( T_\alpha \) and \( T_\gamma \), respectively.

(iii) Let \( \gamma = \alpha \beta, l = k \). According to (13), (14) and the results above (ii), we have

\[
m = (g \sigma_{\alpha,\beta})(h \sigma_{\alpha,\beta}), n = (\lambda \theta_{\alpha,\alpha}))(\mu \theta_{\alpha,\beta}).
\]  
(16)

\[
(i, g, \lambda)(j, h, \mu) = (k, g \circ h, \lambda \circ \mu)
\]  
(17)

According to (i), (ii) and (iv), we know that \( T = [Y : T_\alpha, \sigma_{\alpha,\beta}] \) is a strong semilattice of left\-\(-\)C cancellative monoid \( T_\alpha \) and \( \Lambda = [\Lambda_\alpha : \theta_{\alpha,\beta}] \) is a strong semilattice of right\-\(\)C \(h\)-wpp semigroup \( \Lambda_\alpha \), that is, \((T, \circ)\) is a \(C\)-wpp semigroup and \((\Lambda, \circ)\) is a right normal band. It follows that

\[
(i, g, \lambda)(j, h, \mu) = (k, g \circ h, \lambda \circ \mu)
\]  
(20)

by (18)-(20).

Step 2 We shall show that \( S_\alpha = \cup \alpha \in Y (I_{\alpha} \times T_\alpha) \) forms a left \( C\)-wpp semigroup. Let \( I = \cup \alpha \in Y I_\alpha \). We wish to define a mapping \( \eta : S_\alpha \rightarrow T(Y) \) so that \( S_\alpha \) can be made into a left\-\(\)C semigroup. For all \( k \in I_{\alpha} \), we have

\[
(k, m, n) = (k, m, n)(k^\prime, 1_{\beta}, n) = (i, g, \lambda)(j, h, \mu)(k^\prime, 1_{\beta}, n)
\]  
(18)

\[
(i, g, \lambda)(j, h, \mu) = (k, g \circ h, \lambda \circ \mu)
\]  
(19)

We denote this mapping as \( \phi_{\alpha,\beta}(i, g) \phi_{\alpha,\beta}(j, h) \) is a constant mapping on \( \Lambda_\alpha \), write as \( k = \phi_{\alpha,\beta}(i, g) \phi_{\alpha,\beta}(j, h) \) >, we have

\[
(k, m, n) = (k, m, n)(j, 1_{\beta}, n)(k^\prime, 1_{\beta}, n)
\]  
(20)

\[
(i, g, \lambda)(j, h, \mu)(k^\prime, 1_{\beta}, n) = (i, g, \lambda)(j, h, \mu)(k^\prime, 1_{\beta}, n)
\]  
(21)

\[
\phi_{\alpha,\beta}(i, g) \phi_{\alpha,\beta}(j, h)k^\prime, 1_{\beta}, n)
\]  
(22)

Thus \( k = \phi_{\alpha,\beta}(i, g) \phi_{\alpha,\beta}(j, h) >, \) does not depend on the choice of \( h \), let \( k = \eta(i, g)j \). We define the mapping \( \eta \) by the following rules:

\[
\eta(i, g) : S_\alpha \rightarrow T(I, j), (i, g) \rightarrow \eta(j, g)
\]  
(23)

\[
\eta(i, g) : I \rightarrow I, j \rightarrow \eta(i, g)j
\]  
(24)
and such that
\[(i, g, \lambda)(j, h, \mu) = (\eta(i, g), g \circ h, \lambda \circ \mu)\]
for \((i, g, \lambda), (j, h, \mu) \in S.

To see that \(\eta\) is a structure mapping defining a semi-spined product \(I \times Y T\), we need to verify that \(\eta\) satisfies the required conditions (Q1)-(Q3). If \((i, g) \in I_\alpha \times T_{\alpha, j} \in I_\beta, \alpha \leq \beta,\) then \(\eta(i, g)j = (\phi_{\alpha, \beta}(i, g), \phi_{\alpha, \beta}(j, 1\alpha)) \in I_{\beta},\) (Q1) holds. To verify that (Q2) holds, we let \((i, g) \in I_\alpha \times T_{\alpha, j} \in I_\beta, \alpha \leq \beta,\) then we obtain
\[
(\eta(i, g)j, g \circ h, \lambda \circ \mu) = (i, g, \lambda)[(i, 1\alpha, \lambda)(j, h, \mu)]
\]
by (11) and (20). Consequently, we have
\[
\eta(i, g)j = i.\]

Also prove that \(ker\) by (11) and (20). Thus, \(\eta\) satisfies (Q1)-(Q3) and we do have a semi-spined product \(I \times Y T\).

Next we need to prove that the structure mapping \(\eta\) on this semispined product satisfies the condition (Q) in lemma 1. For all \(\gamma \in Y, l \in I_\gamma, v \in \Lambda_\gamma,\) according to (20), we have
\[
(\eta(\gamma(i, g), g \circ h, \lambda \circ \mu)l = (i, g, \lambda)(j, h, \mu)(l, 1\gamma, v)
\]
This leads to \(\eta(\gamma(i, g)j, g \circ h, \lambda \circ \mu)l = (i, g, \lambda)(j, h, \mu)\). Hence we have
\[
(\eta(i, g)j, g \circ h, \lambda \circ \mu)l = (i, g, \lambda)(j, h, \mu)(l, 1\gamma, v).
\]

Corollary 1 Let \(S\) be a semigroup. Then the following conditions are equivalent:

1. \(S\) is a weakly left C-rpp semigroup;
2. \(S\) is a semilattice of left cancellative monoids, and \(Reg S\) is a weakly left C-semigroup;
3. \(S\) is a semilattice of left cancellative monoids, and \(S\) is a left quasi-normal band;
4. \(S\) is a spined product of left C-rpp semigroup and a right normal band.

Corollary 2 A weakly left C-rpp semigroup is a wrpp semigroup.

Proof. According to theorem 1, a weakly left C-rpp semigroup is a spined product of a left C-rpp semigroup and right normal band, but a left C-rpp semigroup and a right normal band are wrpp semigroups, it follows that a weakly left C-rpp semigroup is a wrpp semigroup.

By above corollary, we have the following results:

Corollary 3 A weakly left C-rpp semigroup is a rpp semigroup.

Corollary 4 A semigroup \(S\) is a weakly left C-semigroup if and only if \(S\) is a spined product of left C-semigroup and a right normal band.

ACKNOWLEDGMENT

This research is supported by Foundation of Shandong Province Natural Science (Grant No.ZR2010AL004).

REFERENCES