Another Structure of Weakly Left C-wrpp Semigroups

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Abstract—It is known that a left C-wrpp semigroup can be described as a curler structure of a left band and a C-wrpp semigroup. In this paper, we introduce the class of weakly left C-wrpp semigroups which includes the class of weakly left C-rpp semigroups as a subclass. We shall particularly show that the spined product of a left C-wrpp semigroup and a right normal band is a weakly left C-wrpp semigroup. Some equivalent characterizations of weakly left C-wrpp semigroups are obtained. Our results extend that of left C-wrpp semigroups.

Keywords—Left C-wrpp semigroup, left quasi normal regular band, weakly left C-wrpp semigroup.

I. INTRODUCTION

THROUGHOUT this paper, we adopt the notation and terminologies given by Howe[1] and Du[2].

Tang[3] considered a Green-like right congruence relation $L^*$ on a semigroup $S$ for $a, b \in S$, $aL^*b$ if and only if $axR_y b$ for all $x, y \in S^1$. Moreover, Tang pointed out in [3] that a semigroup $S$ is a wrpp semigroup if and only if each $L^*$-class of $S$ contains at least one idempotent.

Recall that a wrpp semigroup $S$ is a C-wrpp semigroup if and only if each $L^*$-class of $S$ contains at least one idempotent. A semigroup $S$ is a wrpp semigroup if and only if $S$ is a strong semilattice of left-R cancellative monoids[see[3]]

Because a Clifford semigroup can be expressed as a strong semilattice of groups and a C-rpp semigroup can be expressed as a strong semilattice of left cancellative monoids[see[4-9]], we see immediately that the concept of semigroups is a common generalization of Clifford semigroups and C-rpp semigroups.

For wrpp semigroups, Du-Shum [2] first introduced the concept of left C-wrpp semigroups, that is, a left C-wrpp semigroup whose following conditions: (i) for all $a \in E(L^*_a)$, $a = ac$, where $E(L^*_a)$ is the set of idempotents in $L^*_a$; (ii) for all $a \in S$, there exists a unique idempotent $e \in S$ which is the smallest left *-ideal of $S$ generated by $a$. For such semigroups, Du-Shum[2] gave a method of construction.

Zhang[10] showed that the spined product of a left C-wrpp semigroup and a right normal band which is a weakly left C-wrpp semigroup by virtue of left C-full Ehremann cybergroups. In this paper, we first define the concept of weakly left C-wrpp semigroups. A equivalent descriptions of weakly left C-wrpp semigroups is therefore obtained and our results generalize that of Cao on weakly left C-rpp in[5]. In view of the theorems given in this paper, one can easily observe that the results of weakly left C-wrpp semigroups are a common generalizations of weakly left C-semigroups and left C-wrpp semigroups in range of wrpp semigroups.

II. PRELIMINARIES

We first recall some known results used in the sequel. To start with, we introduce the concept of siemi-spined product.

Let $T = \cup_{\alpha \in Y} T_{\alpha}$ and $I = \cup_{\alpha \in Y} I_{\alpha}$ be the semilattice decomposition of the semigroups $T$ and $I$ with respect to semilattice $Y$ respectively. For all $\alpha \in Y$, we denote the direct product $I_{\alpha} \times T_{\alpha}$ by $S_{\alpha}$. Let $S = \cup_{\alpha \in Y} S_{\alpha}$, we define the mapping $\eta$ by the following rules:

$\eta : S \to T(I), (i, a) \mapsto \eta(i, a), \eta(i, a) : I \to J, j \mapsto (i, a)^* j$

where $T(I)$ is a left transformation semigroup on $I$. Suppose that the mapping $\eta$ satisfies the following conditions:

(Q1) If $(i, a) \in S_{\alpha}, j \in I_{\beta}$, then $(i, a)^* j \in I_{\alpha \beta}$;
(Q2) If $(i, a) \in S_{\alpha}, (j, b) \in S_{\beta}$ with $\alpha \leq \beta$, then $(i, a)^*j = ij$, where $ij$ is the semigroup product in the semigroup $I = \cup_{\alpha \in Y} I_{\alpha}$.
(Q3) If $(i, a) \in S_{\alpha}, (j, b) \in S_{\beta}$, then $\eta(i, a) \eta(j, b) = \eta(i, a)^* j, ab)$, where $ab$ is the semigroup product in the semigroup $T = \cup_{\alpha \in Y} T_{\alpha}$.

Then we define a multiplication $o^*$ on $S = \cup_{\alpha \in Y} S_{\alpha}$ by $(i, a) o^* (j, b) = ((i, a)^* j, ab)$. By a straightforward verification, we can prove that the multiplication $o^*$ satisfies the associative law and hence $(S, o)$ becomes a semigroup, denoted by $S = I \times_{\eta} T$. We call this semigroup the semi-spined product of $I$ and $T$ with respect to the structure mapping $\eta$.

Lemma 1[2] Let $I$ be a left regular band which is expressed as a semilattice of left zero bands $I_{\alpha}$ (that is, $I = \cup_{\alpha \in Y} I_{\alpha}$) and let $T = \cup_{\alpha \in Y} T_{\alpha}$ be a C-wrpp semigroup (that is, $T$ is a strong semilattice of left-R cancellative monoids $Y: T_{\alpha}, (\alpha \in Y)$). If the structure mapping $\eta$ satisfies the following conditions:

(Q): $\ker\eta(i, a) = \ker\eta(j, b)$ for every $(i, a), (j, b) \in S_{\alpha}$.

Then $S$ is a C-wrpp semigroup. Conversely, every left C-wrpp semigroup $S$ can be constructed in terms of above method.

Lemma 2[5] A semigroup $S$ is a weakly left C-semigroup, that is, $S$ is a regular semigroup and

$$\forall e \in E(S), \eta(e) : S \to eS, x \mapsto ex$$

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is a homomorphism if and only if $S$ is a completely regular and $E(S)$ is a left quasi-normal band.

**Lemma 3** [2] If $S$ is a left C-wrpp, then $\text{Reg}S$ is a left C-semigroup.

**Lemma 4** [7] A band $B$ is a left normal band (that is, a band satisfies identity $efg = efg$) if and only if Green relation $L$ and $R$ are congruence on $B$ and $B/R$ is a right normal band.

**Definition 1** A monoid $T$ is a left-regular cancellative monoid if for $a,b,c \in T$, $(ab,ac) \in R$ implies $(b,c) \in R$. We call the direct product of a left-regular cancellative monoid $T$ and a rectangular band $I$ a left cancellative plank because the direct product looks like a two-dimensional plank. We denote the left-regular cancellative plank by $I \times T$.

**Lemma 5** [2] Let $I = \cup_{i \in I} I_i$ be a semilattice of left zero bands, and $T = [Y : T_i, \phi_{a,b}]$ a strong semilattice of left-$R$ cancellative monoids $T_i$. Then $(i,a)R(j,b)$ if and only if $aRb$ and $i = j$ for any $(i,a), (j,b) \in S = \cup_{i \in I} (I_i \times T_i)$.

### III. The weakly left C-wrpp semigroups

In this section, the concept of weakly left C-wrpp semigroups is introduced. We shall give equivalent characterization for the structure of weakly left C-wrpp semigroups. First, we introduce the concept of weakly left C-wrpp semigroups.

**Definition 2** A semigroup $S$ is called a weakly left C-wrpp semigroup, if $S$ is isomorphic to a semilattice of left-$R$ cancellative planks, and

$$(Y \in E(S))\eta_i : S \rightarrow eS, x \mapsto ex$$

is a homomorphism.

We now characterize the weakly left C-wrpp semigroups.

**Theorem 1** Let $S$ be a semigroup. Then the following conditions are equivalent:

1. $S$ is a weakly left C-wrpp semigroup;
2. $S$ is a semilattice of left-$R$ cancellative planks, and $\text{Reg}S$ is a weakly left C-semigroup;
3. $S$ is a semilattice of left-$R$ cancellative planks, and $E(S)$ is a left quasi-normal band;
4. $S$ is a spined product of left C-wrpp semigroups and a right normal band.

**Proof.** (1)$\Rightarrow$(2). We only need show that $\text{Reg}S$ is a weakly left C-semigroup. Let $a,b \in \text{Reg}S$. Then there exists $x, y \in S$ such that $a = axa, x = xax, b = byb$. So $e = xa \in E(S)$. According to (1), we know that $\eta_i$ is a semigroup homomorphism from $S$ to $eS$. Thus

$$ab = axaybab = axa(byb)ba = a(ax)eta_i(b) = (ab)y(g)(ab)$$

So $ab \in \text{Reg}S$. Therefore, $\text{Reg}S$ is a subsemigroup of $S$. Again $E(\text{Reg}S) = E(S)$, according to Lemma 3, we obtain $\text{Reg}S$ is a weakly left C-semigroup.

$(2)\Rightarrow(3)$. Clearly, we omit it.

$(3)\Rightarrow(4)$. Let $S = \cup_{i \in I} (I_i \times T_i \times \Lambda)$ be a semilattice decomposition $Y$ of left-$R$ cancellative planks, and $E(S)$ is a left quasi-normal band, and put $S_1 = \cup_{i \in I} (I_i \times T_i), \Lambda = \cup_{i \in I} \Lambda_i, S_2 = I_i \times T_i \times \Lambda$, where $I_i, T_i$, and $\Lambda_i$ are a left zero band, a left-$R$ cancellative monoid and a right zero band, respectively. Next, we verify that $S_1 = \cup_{\alpha \in \Lambda} (I_i \times T_i)$ is a left C-wrpp semigroup, and $\Lambda = \cup_{\alpha \in \Lambda} \Lambda_i \alpha$ is a right normal band.

Step 1 Let $T = \cup_{\alpha \in \Lambda} T_\alpha$, we shall show that $T$ is a C-wrpp semigroup, and $\Lambda = \cup_{\alpha \in \Lambda} \Lambda_i \alpha$ is a right normal band. For this purpose, we only need to show that $T$ is a strong semilattice of left-$R$ cancellative monoids $T_\alpha$, and a strong semilattice of right zero bands $\Lambda_i$, respectively.

Identity in $T_\alpha$ denoted by $I_\alpha$, obviously, we have $E(S) = \{(i,1,\alpha), (i,1,\lambda) \in I_\alpha \times \lambda, \alpha \in \gamma\}$, and

$$\begin{align*}
(i,1,\alpha) L^E(j,1,\beta) & \iff \alpha = \beta, \lambda = \mu, \\
(i,1,\lambda) R^E(j,1,\mu) & \iff \alpha = \beta, i = j
\end{align*}$$

where $L^E$ and $R^E$ are Green's relations on semigroup $E(S)$. For all $\alpha \geq \beta$, let $a = (i,g,\lambda) \in S_\alpha$, if $(j,\mu) \not\in I_\beta \times \Lambda_\beta$, then there exists $(j_1,h_1,\mu_1) \in S_\beta$ such that $(j,\mu)a = (j_1,h_1,\mu_1)$. Since $(j_1,h_1,\mu_1) = (j_1,\lambda,\mu)(j_1,\mu,\mu_1) = (j_1,\mu_1,\mu_1)$, we get $j_1 = j$. On the other hand, for all $j_1 \in I_\beta$, we have $(j_1,\mu,\mu) = (j_1,\lambda,\mu)(j_1,\mu,\mu) = (j_1,\mu_1,\mu_1)$. So $h_1, \mu_1$ do not depend on the choice of $j$ in $I_\beta$. Let $h_1 = \mu(i,\gamma,\lambda) \chi, \mu_1 = \mu(i,\gamma,\mu) \psi_\alpha$. Therefore, we have

$$(j,\gamma,\lambda)(i,\gamma,\mu) = (\phi_{\beta,\alpha},i,\gamma,\lambda)j, \varphi_{\beta,\alpha}(i,\gamma,\mu)j = \varphi_{\beta,\alpha}(i,\gamma,\mu)j.$$ 

Similarly, we may obtain

$$\begin{align*}
\phi_{\beta,\alpha}(i,\gamma,\lambda)j & = \phi_{\beta,\alpha}(i,\gamma,\mu)j, \\
\varphi_{\beta,\alpha}(i,\gamma,\mu)j & = \varphi_{\beta,\alpha}(i,\gamma,\mu)j,
\end{align*}$$

where $\lambda \in \Lambda_\alpha, \alpha \geq \beta$. Similarly, by $L^E$ is a congruence on $E(S)$, we follow that $\mu(i,\gamma,\lambda) \chi_\beta, \mu(i,\gamma,\lambda) \psi_\beta$ do not depend on the choice of $i$ in $I_\alpha$, let

$$\mu(g,\lambda)X_\beta = \mu(g,\lambda)X_\alpha, \mu(g,\lambda) = \mu(g,\lambda)\psi_\alpha,\beta$$

where $i \in I_\alpha, \alpha \geq \beta$. It follows that $(j,\gamma,\lambda)(i,\gamma,\mu) = (\phi_{\beta,\alpha},i,\gamma,\lambda)j, \varphi_{\beta,\alpha}(i,\gamma,\mu)j = \varphi_{\beta,\alpha}(i,\gamma,\mu)j$. So $\mu(g,\lambda)X_\beta = \mu(g,\lambda)\psi_\alpha,\beta = \mu(g,\lambda)\psi_\alpha,\beta$, write as $c$. Clearly, $c$ is determined by $g$ but does not depend on the choice of $i, j, \lambda$ and $\mu$. Let

$$g_{\sigma,\beta}\sigma = \mu(g,\lambda)\chi_\beta, \sigma = \varphi_{\beta,\alpha}(i,\gamma,\mu)j, (7)$$

where $i \in I_\alpha, j \in I_\beta, \lambda \in \Lambda_\alpha, \mu \in \Lambda_\beta$. According to $L^E$ being a right normal band congruence on $E(S)$, for all $\mu, \mu' \in \Lambda_\beta$, we have $(j,\gamma,\lambda)(i,\gamma,\mu)L^E(\mu',i,\gamma,\lambda)$, that is, $(j,\gamma,\lambda)(i,\gamma,\lambda) \subseteq (j,\gamma,\lambda)(i,\gamma,\lambda)$, we can follow that $(j,\gamma,\lambda)(i,\gamma,\lambda) = (j,\gamma,\lambda)(i,\gamma,\lambda)$ in view...
of (1) and (3), multiplied with \((i, g, \lambda)\) from right side of above formula’s both sides, referring to (3) and (6), we obtain

\[
\mu(g, h)\psi_{\alpha, \beta} = \mu(g, h)\psi_{\alpha, \beta}.
\]

Therefore, \(\mu(g, h)\psi_{\alpha, \beta}\) does not depend on the choice of \(\mu\) in \(\Lambda_\beta\), let

\[
(\phi, \mu) \mapsto \mu(g, h)\psi_{\alpha, \beta}.
\]

where \(\mu \in \Lambda_\beta, \alpha \geq \beta\). In view of (3)-(8), we have

\[
(\phi, (1, g, \mu)) = (g, \psi_{\alpha, \beta}).
\]

Therefore

\[
g_{\alpha, \beta} = (g_{\alpha, \beta}(1), \lambda_{\alpha, \beta}, \psi_{\alpha, \beta}).
\]

Since \(T_\beta\) is a left-\(R\) cancellative monoid,

\[
1_\alpha \sigma_{\alpha, \beta} R_\beta 1_\beta(\alpha \geq \beta),
\]

let

\[
\lambda_{\alpha, \beta} = (1_\alpha, \psi_{\alpha, \beta})(g, \psi_{\alpha, \beta}(g \in T_\alpha, \alpha \geq \beta)).
\]

Thus, summing up the above cases, we conclude that there exists the mapping \(\phi_{\alpha, \beta} : I_\alpha \times T_\alpha \rightarrow T_\beta(1, g, \mu) \mapsto (i, g, \lambda)(i, g, \lambda) \in \Lambda_\alpha \times T_\beta, \lambda \mapsto \lambda_{\alpha, \beta}\) such that

\[
(j, 1, \mu)(i, g) = (i, g, \phi_{\alpha, \beta}(i, g), j, g, \psi_{\alpha, \beta}, \mu, \lambda_{\alpha, \beta}).
\]

for all \((i, g, \lambda) \in S_{\alpha}, (j, \lambda) \in I_{\beta} \times \Lambda_{\beta}\).

The following we verify that \(\sigma_{\alpha, \beta}\) and \(\theta_{\alpha, \beta}\) are the structure homomorphism of strong semilattices on \(T_\alpha\) and \(T_\beta\), respectively. For all \(\alpha, \beta \in Y, (i, g, \lambda) \in S_{\alpha}, (j, h, \mu) \in S_{\beta},\) let \((k, m, n) = (i, g, \lambda)(j, h, \mu) \in S_{\alpha, \beta}\). Then for \(\alpha \leq \gamma \leq \beta\) and \((I, v) \in I_{\gamma} \times \Lambda_{\gamma}\), according to (11), we have

\[
(l, \nu, \sigma_{\alpha, \beta}, n, \sigma_{\alpha, \beta}, \gamma) = (l, 1, v)(k, m, n).
\]

Therefore,

\[
\sigma_{\alpha, \beta} = (g_{\sigma_{\alpha, \beta}}(h_{\sigma_{\alpha, \beta}}), n_{\sigma_{\alpha, \beta}} = (\lambda_{\sigma_{\alpha, \beta}})(\mu_{\sigma_{\alpha, \beta}}),\]

where \(g, h \in T_\alpha, \lambda, \mu \in \Lambda_\alpha\). So \(\sigma_{\alpha, \beta}\) and \(\theta_{\alpha, \beta}\) are semigroup homomorphism of from \(T_\alpha\) to \(T_\beta\) and from \(\Lambda_\alpha\) to \(\Lambda_\beta\), respectively, where \(\alpha \geq \gamma\). Similarly, it follows that \(\sigma_{\alpha, \beta}\) is also a semigroup homomorphism, by (9), we have

\[
1_\alpha \sigma_{\alpha, \beta} = 1_\beta(\alpha \geq \beta).
\]

If \(\beta = \alpha\), let \(\gamma = \alpha, h = 1_\alpha, \mu = \lambda\). In view of (14) and (15), it follows that \(g = \sigma g_{\alpha, \alpha} \in \Lambda_{\alpha, \alpha}\) for any \(g \in T_\alpha, \lambda \in \Lambda_\alpha\). So \(\sigma_{\alpha, \alpha}\) and \(\theta_{\alpha, \alpha}\) are identical mapping on \(T_\alpha\) and \(\Lambda_\alpha\), respectively.

Let \(\gamma = \alpha, \mu = k\). According to (13), (14) and the results above, we have

\[
m = (g_{\sigma_{\alpha, \alpha}}(h_{\sigma_{\alpha, \beta}}), n = (\lambda_{\sigma_{\alpha, \alpha}})(\mu_{\sigma_{\alpha, \beta}}),\]

(iii) Let \(\gamma = \alpha, \beta, l = k\). According to (13), (14) and the results above, we have

\[
m = (g_{\sigma_{\alpha, \alpha}}(h_{\sigma_{\alpha, \beta}}), n = (\lambda_{\sigma_{\alpha, \alpha}})(\mu_{\sigma_{\alpha, \beta}}),\]

(16)

(17)

(18)

(19)

(e) If \(\alpha \geq \beta \geq \gamma\), then \(\alpha \beta = \beta\). Referring to (13), (16) and (17), we have \(g_{\sigma_{\alpha, \alpha}}(h_{\sigma_{\alpha, \beta}}) = (g_{\sigma_{\alpha, \alpha}}(h_{\sigma_{\alpha, \beta}}), n = (\lambda_{\sigma_{\alpha, \alpha}})(\mu_{\sigma_{\alpha, \beta}}),\)

This leads to \(\sigma_{\alpha, \beta} = \sigma_{\alpha, \beta} \theta_{\alpha, \beta} = \sigma_{\alpha, \beta}\). Define multiplication operations on \(T = \bigcup_{\alpha \in \gamma} T_\alpha\) and \(\Lambda = \bigcup_{\alpha \in \gamma} \Lambda_\alpha\), as follows respectively:

\[
g \circ h = (g_{\sigma_{\alpha, \alpha}}(h_{\sigma_{\alpha, \beta}}), n = (\lambda_{\sigma_{\alpha, \alpha}})(\mu_{\sigma_{\alpha, \beta}}),\]

(18)

(19)

(20)

by (18)-(20).

Step 2 We shall show that \(S_t = \bigcup_{\alpha \in \gamma} (I_\alpha \times T_\alpha)\) forms a left C-wrpp semigroup. Let \(I = \bigcup_{\alpha \in \gamma} I_\alpha\). We wish to define a mapping \(\eta : S_t \rightarrow T_\gamma\) so that \(S_t\) can be made into a semi-group product. For all \(k \in I_{\alpha, \beta}\), we have

\[
(k, m, n) = (k, m, n)(1, \beta, n) = (i, g, \lambda)(j, h, \mu)(k', 1, \alpha, n)\]

\[
= (i, g, \lambda)(\phi_{\alpha, \beta}(j, h), k', \ldots).
\]

So \(k = \phi_{\alpha, \beta}(i, g)\phi_{\alpha, \beta}(j, h)\). Therefore, \(\phi_{\alpha, \beta}(i, g)\phi_{\alpha, \beta}(j, h)\) is a constant mapping on \(I_{\alpha, \beta}\), write as \(k = \phi_{\alpha, \beta}(i, g)\phi_{\alpha, \beta}(j, h)\), we have

\[
(k, m, n) = (k, m, n)(1, \beta, n)(k', 1, \alpha, n)\]

\[
= (i, g, \lambda)(j, h, \mu)(\phi_{\alpha, \beta}(j, h), k', \ldots).\]

\[
= \phi_{\alpha, \beta}(i, g)\phi_{\alpha, \beta}(j, h)(j, h, k', \ldots).
\]

Thus \(k = \phi_{\alpha, \beta}(i, g)\phi_{\alpha, \beta}(j, h)\) does not depend on the choice of \(h\), let \(k = \eta(i, j)\). We define the mapping \(\eta\) by the following rules:

\[
\eta(i, g) : S_t \rightarrow T(I, (i, g) \rightarrow \eta(j, g);\]

\[
\eta(i, g) : I \rightarrow I, j \rightarrow \eta(i, g)j.
\]
and such that
\[(i, g, \lambda)(j, h, \mu) = (\eta(i, g), g \circ h, \lambda \circ \mu)\]
for \((i, g, \lambda), (j, h, \mu) \in S\).

To see that \(\eta\) is a structure mapping defining a semi-spined product \(I \times \Lambda \times T\), we need to verify that \(\eta\) satisfies the required conditions (Q1)-(Q3). If \((i, g) \in I_\alpha \times T_\alpha, j \in I_\beta, \alpha \leq \beta\), then
\[\eta(i, g)j = \eta(i, g)(\alpha, \beta, j, 1_\alpha, 1_\beta) \in I_{\alpha, \beta}\]
holds. To verify that (Q2) holds, we let \((i, g) \in I_\alpha \times T_\alpha, j \in I_\beta, \alpha \leq \beta\), then we obtain
\[(\eta(i, g)j, g \circ h, \lambda \circ \mu) = (i, g, \lambda)(\eta(j, h), h, \mu)\]
by (11) and (20). Consequently, we have \(\eta(i, g)j = i\). Thus, (Q2) holds. Finally, we let \((i, g) \in I_\alpha \times T_\alpha, (j, h) \in I_\alpha \times T_\beta\). For all \(\gamma \in \Lambda, l \in I_\gamma, \alpha \in \Lambda\), according to (20), we have
\[(\eta(i, g)j, g \circ h), (g \circ h, 1_\gamma) \in \gamma, \lambda \circ \mu\]
This leads to \(\eta((i, g)j, g \circ h) = \eta(i, g)((j, h), h, \mu)\), so \(\eta(i, g)j, g \circ h = \eta(i, g)\eta(j, h)\). In fact, we have shown that (Q3) holds. Thus, \(\eta\) satisfies (Q1)-(Q3) and we do have a semi-spined product \(I \times \Lambda \times T\).

Next we need to prove that the structure mapping \(\eta\) on this semifield of \((i, g, \lambda)(j, h, \mu)\) satisfies the condition (Q) in lemma 1. For this purpose, we let \((i, a) \in I_\alpha \times T_\alpha\), \((j, b) \in I_\beta \times T_\beta\). Take \(k \in I_\tau\) and \(l \in I_\gamma\) for some \(\tau, \gamma \), and suppose that \(\eta(i, a)k = \eta(j, b)l\), that is, \(\alpha, \alpha, \beta, \gamma, \lambda \circ \mu\). By condition (Q1), we have \(\delta \alpha = \tau\alpha\). Denote the identity elements of the monoids \(I_\tau\) and \(T_\gamma\) by \(1_\tau\) and \(1_\gamma\), respectively. Since \(T\) is a strong semilattice of \(T_\alpha\), we have \(a_1 \alpha = a_1\). By invoking Lemma 5, we have \((i, a)(k, 1_\beta) = (i, a)(1_\alpha, 1_\beta)\). Since \(i \gamma, j\), we have \((i, a)C^*(j, b)\) so that \((j, b)(k, 1_\gamma) = (j, b)(1_\alpha, 1_\gamma)\). Hence we have
\[(j, b)(k, 1_\gamma)R((j, b)(k, 1_\gamma)) \Rightarrow (j, b)(k, 1_\gamma)k = (j, b)k = (j, b)k\]
This shows that \(\kappa(i, a) \subseteq \kappa(j, b)\). Analogously, we can also prove that \(\kappa(j, b) \subseteq \kappa(i, a)\). Thus \(\kappa(i, a) = \kappa(j, b)\) and so condition (Q) is satisfied. This shows that \(S_\tau \subseteq \cup_{\alpha \in \Lambda}(I_\alpha \times T_\alpha)\) is indeed a left C-wrpp semigroup.

Summing step 1 and step 2, we conclude that \(S\) is the spined product of a left C-wrpp semigroup \(S_1\) and a right normal band \(\Lambda\).

Let \(S\) be the spined product of a left C-wrpp semigroup \(S_1 = I \times \Lambda \times T\) and a right normal band \(\Lambda = [Y; \Lambda_0, \theta, \lambda, \rho]\). Clearly, \(S\) is a semilattice of right-C cancellative planks, and for all \(e = (i, 1_\alpha, 1_\beta) \in E(S) \cap (I_\alpha \times T_\alpha \times \Lambda_0), x = (j, h, \mu) \in I_\gamma \times T_\gamma \times \Lambda_\gamma, \gamma = (k, m, n) \in I_\gamma \times T_\gamma \times \Lambda_\gamma\), let \((l, q) = (i, 1_\alpha)(j, h) \in I_{\alpha, \beta} \times T_{\alpha, \beta}\). According to \(S_1\) is a left C-wrpp semigroup and Lemma 1, we have \((i, q)(i, 1_\alpha) = (\eta(i, g)j, g \circ h, \lambda \circ \mu)\). Consequently, \(\eta\) is a semigroup homomorphism from \(S\) to \(eS\), thus \(S\) is a weakly left C-wrpp semigroup.

**Corollary 1** Let \(S\) be a semigroup. Then the following conditions are equivalent:
(1) \(S\) is a weakly left C-rpp semigroup;
(2) \(S\) is a semilattice of left cancellative monoids, and \(\text{Reg}S\) is a weakly left C-semigroup;
(3) \(S\) is a semilattice of left cancellative monoids, and \(S\) is a left quasi-normal band;
(4) \(S\) is a spined product of left C-wrpp semigroup and a right normal band.

**Corollary 2** A weakly left C-wrpp semigroup is a wrpp semigroup.

Proof. According to theorem 1, a weakly left C-wrpp semigroup is a spined product of a left C-wrpp semigroup and right normal band, but a left C-wrpp semigroup and a right normal band are wrpp semigroups, it follows that a weakly left C-wrpp semigroup is a wrpp semigroup.

By above corollary, we have the following results:

**Corollary 3** A weakly left C-rpp semigroup is a rp pp semigroup.

**Corollary 4** A semigroup \(S\) is a weakly left C-semigroup if and only if \(S\) is a spined product of a left C-semigroup and a right normal band.

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