Performance Analysis of MUSIC, Root-MUSIC and ESPRIT DOA Estimation Algorithm

N. P. Waweru, D. B. O. Konditi, P. K. Langat

Abstract—Direction of Arrival estimation refers to defining a mathematical function called a pseudospectrum that gives an indication of the angle a signal is impinging on the antenna array. This estimation is an efficient method of improving the quality of service in a communication system by focusing the reception and transmission only in the estimated direction thereby increasing fidelity with a provision to suppress interferers. This improvement is largely dependent on the performance of the algorithm employed in the estimation. Many DOA algorithms exist amongst which are MUSIC, Root-MUSIC and ESPRIT. In this paper, performance of these three algorithms is analyzed in terms of complexity, accuracy as assessed and characterized by the CRLB and memory requirements in various environments and array sizes. It is found that the three algorithms are high resolution and dependent on the operating environment and the array size.

Keywords—Direction of Arrival, Autocorrelation matrix, Eigenvalue decomposition, MUSIC, ESPRIT, CRLB.

I. INTRODUCTION

WIRELESS communication is one of the fastest growing fields in the engineering world. This has been necessitated by the advancement made in the research and design of communication equipment [1], [2]. It started with cellular communication, then the world-wide-web to the extent where these services are accessible wherever whenever. The Quality of Service (QoS) however, deteriorates with distance between the transmitting node and the receiver.

Estimating the direction of a transceiver is an effective method of improving the QoS between a node and a transceiver, by heightening and focusing the transmission only to the direction of the receiver and vice versa for the receiver [3]–[5]. This is achieved by the use antenna arrays with some added capability to estimate the Direction of Arrival (DOA) of all impinging signals.

The aforementioned improvement is to a large extent dependent on the performance of the employed algorithm. The performance of a DOA algorithm is in turn dependent on the size of the array, number of impinging signals, spacing between elements and the number of snapshots used in the estimation process.

Many DOA techniques exist: quadratic type e.g. CAPON and those based on eigenvalue decomposition e.g. MUSIC, ESPRIT and their variants. Eigenvalue decomposition based algorithms involves manipulation of the signal autocorrelation matrix to yield noise and signal subspaces from which angles of arrival of impinging signals are extracted.

This paper therefore, seeks to analyze the performance of three DOA algorithms based on number of array elements and the Signal to Noise Ratio (SNR) to have an optimum choice of algorithm and design for a given environment.

MUSIC, Root-MUSIC and ESPRIT algorithms were developed and simulated in MATLAB software for signals operating at 2.4GHz for a ULA with inter-element spacing of \( \frac{\lambda}{2} \). In Section II of this paper, the signal model of an \( N \) element uniform linear array receiving \( M \) signals from directions \( \theta_1, \theta_2, \ldots, \theta_M \) is derived followed by the Cramer-Rao Lower Bound (CRLB) for DOA estimation in III. Sections IV-VI briefly describe the mathematical aspects behind the three algorithms. Simulation results are presented in Section VII followed by the discussion and conclusion in Sections VIII and IX respectively.

II. SIGNAL MODEL

Consider a uniform linear array (ULA) having \( N \) identical elements, separated by a distance \( d \) as shown in Fig. 1. This array is receiving a far field signal impinging the array at an angle \( \theta \) to the array axis. Taking element 1 as the reference, the line path from the source to the \( i^{th} \) element is shorter than...
that to the 1st element. If the received signal at sensor 1 is \( x_1(t) = s(t) \), it is delayed at sensor \( i \) by
\[
\tau_i = \frac{(i-1)d\sin\theta}{c}
\]
Then the received signal at sensor \( i \) is
\[
x_i(t) = e^{-j\omega\tau_i}x_1(t) = e^{-jkd(i-1)\sin\theta}s(t) = e^{j(i-1)\psi}s(t)
\]
Putting received signals from all \( N \) elements together,
\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-j\psi} \\ \vdots \\ e^{j(N-1)\psi} \end{bmatrix} s(t) = a(\theta)s(t)
\]
where \( a(\theta) \) is called the steering vector.

If there are \( M \) signal sources, received by the array arriving at angles \( \theta_1, \theta_2, \ldots, \theta_M \), we get a signal model
\[
x(t) = As(t)
\]
where the vector \( A \) is the array steering vector having the following Vandermonde structure.
\[
A = \begin{bmatrix} e^{-j2\psi_1} & e^{-j2\psi_2} & \ldots & e^{-j2\psi_M} \\ e^{-j\psi_1} & e^{-j\psi_2} & \ldots & e^{-j\psi_M} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j(N-1)\psi_1} & e^{-j(N-1)\psi_2} & \ldots & e^{-j(N-1)\psi_M} \end{bmatrix}
\]

III. CRAMER-RAO LOWER BOUND

Cramer-Rao Lower Bound provides an algorithm independent means of assessing and comparing the accuracy and performance of a DOA algorithm. The CRBL on the variance of direction estimation errors provides a useful characterization of the achievable accuracy of the DOA system. This is achieved by comparing the Mean Square Error(MSE) with the CRBL [6, 7].

The CRBL theorem states that for a length \( N \) vector of received signal \( x \) dependent on a set of parameters \( P \), and corrupted by additive noise, the variance of an unbiased estimate of the \( p^{th} \) estimate is greater than the cramer-rao lower bound.

The CRBL of a DOA estimation problem is given by
\[
\text{var}(\theta) \geq \text{CRBL} = \frac{6}{\text{var}[N(N^2-1)(kd)^2\sin^2\theta]}
\]

IV. MUSIC

Multiple Signal Classification (MUSIC) is a popular high resolution algorithm based on eigenstructure technique. The main idea behind this DOA algorithm is that of performing eigenvalue decomposition on the correlation matrix [8, 9], separating it into two subspaces: signal subspace and the noise subspace. Since the signal subspace is spanned by the array steering vector of the received signals, this makes the steering vector orthogonal to the noise subspace. The product of the two: array steering vector and the noise subspace, therefore is a null for a particular Angle of Arrival (AOA).

For a uniform linear array with \( N \) elements and \( M \) signals \( s_1(t), s_2(t), \ldots, s_M(t) \) arriving from directions \( \theta_1, \theta_2, \ldots, \theta_M \), and in the presence of noise \( n(t) \), the received signal \( x(t) \) is given by
\[
x(t) = As(t) + n(t)
\]
Defining an \( N \times N \) autocorrelation matrix of the received signal \( R_{xx} \) as
\[
R_{xx} = E\{x(t)x^H(t)\} = AR_{ss}A^H + \sigma^2_0 I
\]
where:
\[
R_{ss} = E\{s(t)s^H(t)\} = \text{diag}\{\sigma^2_1, \ldots, \sigma^2_M\}
\]
\( R_{xx} \) has \( N \) eigenvalues \( \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \) and \( N \) associated eigenvectors making a subspace \( \mathcal{E} = \{\varpi_1, \varpi_2, \ldots, \varpi_N\} \). Sorting the \( N \) eigenvalues from the smallest to the largest, the subspace \( \mathcal{E} \) can be decomposed into two subspaces:
\[
\mathcal{E} = \begin{bmatrix} \varpi_1 & \varpi_2 & \cdots & \varpi_N \end{bmatrix}_{EN} = \begin{bmatrix} E_N & E_S \end{bmatrix}
\]
\( E_N \) is the \( N \times (N - M) \) noise subspace composed of the eigenvectors associated with the noise, whereas \( E_S \) is the \( N \times M \) signal subspace composed of the eigenvectors associated with the arriving signal.

Due to the orthogonality of the noise subspace and the array steering vector at the angles of arrival \( \theta_1, \theta_2, \ldots, \theta_M \), the matrix product \( a^H(\theta)E_NE_N^H a(\theta) \) is zero for this angles. The reciprocal of this matrix product creates sharp peaks at the angle of arrival. Thus the MUSIC pseudospectrum is given as
\[
P(\theta) = \frac{1}{|a^H(\theta)E_NE_N^H a(\theta)|}
\]

V. ROOT-MUSIC

This is a variant of MUSIC algorithm that employs more information than MUSIC [10]. Unlike MUSIC which involves plotting the pseudospectrum against the angles and searching for the peaks, ROOT-MUSIC involves finding the roots of a polynomial.

Starting with the pseudospectrum of MUSIC algorithm
\[
P(\theta) = \frac{1}{|a^H(\theta)C a(\theta)|}
\]
defining \( C = E_NE_N^H \), the denominator of equation 12 above can be rewritten as
\[
P(\theta) = \frac{1}{|a^H(\theta)CA(\theta)|}
\]
the \( m^{th} \) element \( a_m(\theta) \) of the array steering vector is defined as
\[
a_m(\theta) = e^{-jkd_m\sin\theta}, \quad m = 0, 1, \ldots, N - 1
\]
The denominator, thus can be rewritten as

\[ a^H(\theta) C a(\theta) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{-j k d m \sin \theta} c_{mn} e^{j k d n \sin \theta} \]

\[ = \sum_{\ell=-N+1}^{N-1} C_{\ell} e^{j k \ell \sin \theta} \]

(16)

where \( C_{\ell} \) is the sum of the elements along the \( \ell^{th} \) diagonal of \( C \).

Letting \( z = e^{-j k d \sin \theta} \), equation 16 above simplifies to

\[ D(z) = \sum_{\ell=-N+1}^{N-1} C_{\ell} z^{\ell} \]

(17)

The roots of \( D(z) \) that lie closest to the unit circle correspond to the poles of the MUSIC pseudospectrum. These \( 2(N-1) \) roots can be written as

\[ z_i = |z_i| e^{j \arg(z_i)}, \quad i = 1, 2, \ldots, 2(N-1) \]

(18)

Choosing those roots inside the unit circle whose magnitude \( |z_i| \approx 1 \), and comparing \( e^{j \arg(z_i)} \) to \( e^{-j k d \sin \theta} \) gives

\[ \theta_i = -\sin^{-1} \left\{ \frac{\arg z_i}{k d} \right\} \]

(19)

VI. ESPRIT

Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT) algorithm involves decomposing an \( N \) element array into two identical subarrays each with \( S \) element [8], [11]. The objective of ESPRIT algorithm is to estimate the angle of arrival by determining the rotation operator \( \Phi \).

The separation distance between the two subarrays is \( \Delta \) (measured in wavelengths). Fig. 2 show a ten element linear array and possible subarray configurations for \( S=9, 7 \) and \( 5 \) elements respectively \((\Delta=1, 3 \) and \( 5)\). The \( 1^{st} \) element in the subarray is the first element in the first subarray whereas the \((\Delta+1)^{th} \) element of the original sensor is the first element in the second subarray.

Letting

N-Number of elements in the original array
S-Number of elements in each subarray
M-Number of signals hitting the subarray.

Also letting \( x_1(t) \) and \( x_2(t) \) be the received signal in the two subarrays, corrupted by additive white gaussian noise \( n_1(t) \) and \( n_2(t) \) respectively.

\[ x_1(t) = A s(t) + n_1(t) \]

\[ x_2(t) = A \Phi s(t) + n_2(t) \]

(20)

where \( x_1(t), x_2(t), n_1(t) \) and \( n_2(t) \) are \( M \times 1 \) matrices. \( A \) is the \( S \times M \) steering matrix and the variable \( \Phi \) is \( M \times M \) diagonal matrix called the rotation operator.

\[ \Phi = diag\{e^{j \psi_1}, e^{j \psi_2}, \ldots, e^{j \psi_M}\} \]

(21)

where

\[ \psi_i = -2 k \Delta \sin \theta_i; \quad 1 \leq i \leq M \]

(22)

From equation 20, correlation matrices \( R_{11} \) and \( R_{22} \) of the signals in the two subarrays can be estimated as

\[ R_{11} = E\{x_1(t)x_1^H(t)\} \]

\[ R_{22} = E\{x_2(t)x_2^H(t)\} \]

(23)

Eigen-decomposing \( R_{11} \) and \( R_{22} \) result in two signal subspaces \( E_1 \) and \( E_2 \) respectively. Defining a \( 2M \times 2M \) matrix \( C \) from the two subspaces such that

\[ C = \begin{bmatrix} E_1^H & E_2^H \\ E_1 & E_2 \end{bmatrix} \]

(24)

\[ E_1 \]

\[ E_2 \]

\[ C \]

\[ \Phi = -E_{12} E_{22}^{-1} \]

(26)

From \( M \) eigenvalues of \( \Phi \), angles of arrival can be estimated as

\[ \theta_i = \sin^{-1} \left\{ \frac{\arg(\lambda_i)}{k \Delta} \right\} \]

(27)

VII. SIMULATION RESULTS

Simulations were done in MATLAB software for angles of arrival \( \theta_1 = -64^\circ, \theta_2 = 0^\circ, \theta_1 = 23^\circ \) and \( \theta_3 = 58^\circ \) respectively. The array size was held to 8 elements as the values of SNR were varied from 0-100dB in steps of 20dB. This was repeated holding SNR to 50dB and varying the array size from 5-100 elements.
Fig. 3 shows the performance of MUSIC algorithm in various operating conditions. Fig. 3a shows the performance of this algorithm in an environment with varying SNR. For low values of SNR, 0dB, the spikes depicting the arrival of a...
signal from certain direction are small and the response is almost flat. It is thus difficult to exactly extract the angles of arrival. As the values of SNR increase, however, the resolution of the algorithm is observed to improve.
considerably and the spikes become more definite. This is attributed to the fact that for low SNR the difference between the eigenvalues associated with the signal and those associated with the noise become smaller and the peaks...
Therefore, the noise becomes smaller with respect to the noise levels. With increase in SNR, the difference between the two sets of eigenvalues is substantial and the peaks are bigger with respect to the noise levels.

Fig. 3b depicts the response of the MUSIC algorithm to varying number of snapshots. It can be seen from the figure that for 20 snapshots the response has less pronounced spikes. The resolution is seen to improve with increase in the number of snapshots from 20 to 200,000. The number of snapshots affect the correlation between the received signals. For less snapshots, the received signals seem more correlated making it difficult to distinguish between them.

The response of MUSIC algorithm to the array size is as shown in Fig. 3c. From the figure, it can be seen that for 5 elements, the spikes are very definite and exactly correspond the the angles of arrival. As the array size increase, the resolution improves and the extraction of the angle of arrival becomes easier. An extract of the pseudospectra for N=5 and N=200 are shown in Fig. 3d and 3e respectively.

MUSIC pseudospectrum for two closely spaced signals separated by $\theta_2-\theta_1=1^\circ$ and $\theta_2=0^\circ$ in different environments. From this, it is clear that Root-MUSIC is a high resolution DOA algorithm which can estimate two closely spaced angles of arrival as two angles and not one.

Fig. 4e and 4f depicts the accuracy of the Root-MUSIC for a signal impinging on the array specifically at $23^\circ$ for varying values of array size and SNR. It can be observed that the performance of this algorithm is less dependent on the array size but largely on the operating environment but it anyhow improves with the improvement of either.

Tables I and II shows the response of the ESPRIT algorithm to varying values of SNR and array size. As in the other two algorithms, the resolution is very poor for low values of SNR and improves as the SNR values increase. Minimal variation in the accuracy of the algorithm with the array size is observed.

Fig. 5 summarizes the performance of the three algorithms in terms of the Mean Square Error (MSE) and in comparison with CRLB as the values of the SNR and the array sizes are varied. Four angles of arrival $\theta_1=-36^\circ$, $\theta_2=-3^\circ$, $\theta_1=0^\circ$ and $\theta_1=87^\circ$ (two closely spaced and one near endfire). From Fig. 5a-5e, it can be observed that the MSE diminishes with increase in both SNR and array size and the performance approaches the CRLB. However, variation of MSE with the array size is flat and approaching the CRLB behaviour for ESPRIT algorithm as captured in Fig. 5f.

**IX. CONCLUSION**

From the discussion above, where three DOA algorithms are developed in MATLAB and their response to various parameters presented, it can be concluded that the three algorithms are high resolution algorithms. This is derived from the precision with which the angles of arrival are estimated. The three algorithm are highly sensitive to the signal to noise ratio SNR where the resolution of the algorithms is found to improve with the increase in the SNR. Resolution is also found to considerably increase with the number of elements forming the antenna array.

MUSIC algorithms is suitable for an array with few array elements in an environment with high SNR with average snapshots whereas root-MUSIC requires relatively more elements than MUSIC in a high SNR environment again with average snapshots. ESPRIT algorithm on the other hand is less dependent on the array size and can perform relatively well than the other two in an environment having low SNR.

**REFERENCES**


