Some Results on New Preconditioned Generalized Mixed-Type Splitting Iterative Methods

Guangbin Wang, Fuping Tan, Deyu Sun

Abstract—In this paper, we present new preconditioned generalized mixed-type splitting (GMTS) methods for solving weighted linear least square problems. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. Finally, we give a numerical example to confirm our theoretical results.

Keywords—Preconditioned, GMTS method, linear system, convergence, comparison.

I. INTRODUCTION

Sometimes, one has to solve a nonsingular linear system as

$$Hy = f,$$

where

$$H = \begin{pmatrix} I - B & U \\ L & I - C \end{pmatrix}$$

is an invertible matrix with

$$B = \begin{pmatrix} b_{ij} \end{pmatrix}_{p \times p}, \quad C = \begin{pmatrix} c_{ij} \end{pmatrix}_{q \times q}, \quad L = \begin{pmatrix} l_{ij} \end{pmatrix}_{p \times p}, \quad U = \begin{pmatrix} u_{ij} \end{pmatrix}_{p \times q}.$$  

Throughout the paper, we consider the following decomposition for the matrix $H$, $H = D - L - U$, in which

$$D = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix}, \quad U = \begin{pmatrix} B & -U \\ 0 & C \end{pmatrix}.$$  

In [1], authors presented a generalized mixed-type splitting (GMTS) iterative method and preconditioned generalized mixed-type splitting (PGMTS) iterative methods to solve systems of linear equations (1). They showed that the PGMTS methods converge faster than the GMTS method, whenever the GMTS method is convergent.

This paper is organized as follows. In Section II, we give some important definitions and the known results as the preliminaries of the paper. In Section III, we propose three preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. In Section IV, we give an example to confirm our theoretical results.

II. PRELIMINARIES

Definition 1[2] $A \in \mathbb{R}^{n \times n}$ is called a Z-matrix if $a_{ij} \leq 0$ for $i, j = 1, 2, \ldots, n(i \neq j)$.

Definition 2[2] Let $A$ be a Z-matrix with positive diagonal elements. Then the matrix $A$ is called an M-matrix if $A$ is nonsingular and $A^{-1} \geq 0$.

Definition 3[3] The splitting $A = M - N$ is called

(1) a regular splitting of $A$ if $M^{-1} \geq 0$ and $N \geq 0$;

(2) a nonnegative splitting of $A$ if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$;

(3) a weak nonnegative splitting of $A$ if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the first type) or $NM^{-1} \geq 0$ (the second type);

(4) a convergent splitting of $A$ if $\rho(M^{-1}N) < 1$.

Lemma 1[1] Let $A$ be a Z-matrix. Moreover, suppose that $A = M - N$ is a weak nonnegative splitting of the first type. Then $\rho(M^{-1}N) < 1$ if and only if $A$ is an M-matrix.

Lemma 2[4] Let $A = M - N$ be a regular splitting of $A$. Then $\rho(M^{-1}N) < 1$ if and only if $A$ is nonsingular and $A^{-1}$ is nonnegative.

Lemma 3[5] Let matrix $A = (a_{ij})_{n \times n}$ be given such that

(1) $a_{ij} \leq 0$ for $i, j = 1, 2, \ldots, n(i \neq j)$

(2) $A$ is nonsingular.

(3) $A^{-1} \geq 0$.

Then,

(4) $a_{ij} > 0$ for $i, j = 1, 2, \ldots, n$, i.e., $A$ is an M-matrix.

(5) $\rho(B) < 1$ where $B = I - D^{-1}A$, where $D = \text{diag}\{a_{i1}, \ldots, a_{nn}\}$.
Lemma 4 [3] Let \( A = M_1 - N_1 = M_2 - N_2 \) be two convergent weak nonnegative splittings of \( A \), where \( A^{-1} \geq 0 \), if \( M_1^{-1} \geq M_2^{-1} \) then \( \rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) \).

III. THE PRECONDITIONED GMTS METHOD AND THE COMPARISON RESULTS

Consider the linear system (1), the generalized mixed-type splitting (GMST) iterative method is given as follows:

\[
(\hat{D} + D_1 + L_1 - \hat{L})y^{(k+1)} = (D_1 + L_1 + \hat{U})y^{(k)} + f
\]

where \( \hat{D} \), \( \hat{L} \), and \( \hat{U} \) are defined by (2), and \( D_1 \) is an auxiliary nonnegative block diagonal matrix, \( L_1 \) is an auxiliary strictly nonnegative block lower triangular matrix such that \( 0 \leq L_1 \leq \hat{L} \). Evidently, the iteration matrix of the GMS iterative method is given as follow:

\[
T = (\hat{D} + D_1 + L_1 - \hat{L})^{-1}(D_1 + L_1 + \hat{U}).
\]

In this paper, we propose the preconditioners as follow,

\[
P_i^* = \begin{pmatrix} I + S_i & 0 \\ 0 & I + V_i \end{pmatrix}, \quad i = 1, 2,
\]

where

\[
S_i = \begin{pmatrix} 0 & \alpha_1 + b_{12} & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \alpha_p + b_{p,p-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots \\ \end{pmatrix},
\]

\[
V_i = \begin{pmatrix} 0 & \gamma_1 + c_{12} & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \gamma_q + c_{q,q-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots \\ \end{pmatrix},
\]

\[
\hat{D} = \begin{pmatrix} 0 & b_{12} & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots \\ \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} \delta_1 + c_{12} & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots \\ \end{pmatrix},
\]

Let us consider the corresponding splitting for the preconditioned GMS (PGMS) method, that is the generalized mixed-type splitting for the \( M^* = P_i^*H = M_i^* - N_i^* \), where

\[
M_i^* = \hat{D} + D_1 + L_1 - \hat{L}, \quad N_i^* = D_1 + L_1 + \hat{U}^* \text{ and}
\]

\[
\hat{D}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{L}^* = \begin{pmatrix} 0 & 0 \\ -L_* & 0 \end{pmatrix}, \quad \hat{U}^* = \begin{pmatrix} b_{i,j} & -U_{i,j} \\ 0 & C_{i,j} \end{pmatrix}, \quad i = 1, 2.
\]

The iterative matrix of the preconditioned GMS method is

\[
T_i^* = (\hat{D} + D_1 + L_1 - \hat{L})^{-1}(\hat{D} + L_1 + \hat{U}^*).
\]

Now, we show that in the case that the GMS converges, the preconditioned GMS methods converge faster.

Lemma 5 [1] Assume that \( L \leq 0 \), \( U \leq 0 \), \( B \geq 0 \), \( C \geq 0 \) and \( H \) in (1) is irreducible. If \( D_1 \) is nonsingular, then the iterative matrix of the GMS method is reducible.

Lemma 6 [1] Assume that \( L \leq 0 \), \( U \leq 0 \), \( B \geq 0 \), \( C \geq 0 \), then the corresponding splitting of GMS method is a regular splitting for the matrix \( H \).

Similar to the proof of Lemma 6, we can prove the following lemmas.

Lemma 7 Assume that \( L \leq 0 \), \( U \leq 0 \), \( B \geq 0 \), \( C \geq 0 \),

\[
-b_{j-1,i} < \alpha_i < \frac{b_{j-1,i}}{1-b_{ji}}, \quad -c_{j-1,i} < \gamma_j < \frac{c_{j-1,i}}{1-c_{ji}},
\]

for some \( i \in \{2,3,...,p\} \), \( j \in \{2,3,...,q\} \). Then the corresponding splitting of PGMS method is a regular splitting for the matrix \( P_i^*H \).

Lemma 8 Assume that \( L \leq 0 \), \( U \leq 0 \), \( B \geq 0 \), \( C \geq 0 \),

\[
-b_{j-1,i} < \beta_i < \frac{b_{j-1,i}}{1-b_{ji}}, \quad -c_{j-1,i} < \delta_j < \frac{c_{j-1,i}}{1-c_{ji}},
\]

for some \( i \in \{2,3,...,p\} \), \( j \in \{2,3,...,q\} \). Then the corresponding splitting of PGMS method is a regular splitting for the matrix \( P_i^*H \).

Theorem 1 Let \( H \) be an M-matrix and

\[
-b_{j-1,i} < \alpha_i < \frac{b_{j-1,i}}{1-b_{ji}}, \quad -c_{j-1,i} < \gamma_j < \frac{c_{j-1,i}}{1-c_{ji}},
\]

for some \( i \in \{2,3,...,p\} \), \( j \in \{2,3,...,p-1\} \). Then \( P_i^*H \) is an M-matrix.

Proof. Consider the following splitting for \( H \), \( H = M_i - N_i \), where \( M_i = (P_i^*)^{-1} \) and \( N_i = (P_i^*)^{-1}(\hat{L} + \hat{U}^*) \), where

\[
\hat{L} = \begin{pmatrix} 0 & 0 \\ -L_* & 0 \end{pmatrix}, \quad \text{and} \quad \hat{U}^* = \begin{pmatrix} b_{i,j} & -U_{i,j} \\ 0 & C_{i,j} \end{pmatrix}.
\]
It is easy to see that $M_i^{-1}N_i = \hat{L} + \hat{U}$ and $M_i^{-1} \geq 0$. So we can get that $H = M_i - N_i$ is a weak nonnegative splitting of the first type. By the assumption $H$ is an M-matrix, hence Lemma 1 implies that $\rho(M_i^{-1}N_i) < 1$. Let us assume that $P_i' H = I - \hat{L} - \hat{U}$, using the fact that $\rho(\hat{L} + \hat{U}) = \rho(M_i^{-1}N_i) < 1$, by Lemma 2 and Lemma 3, the result follows immediately.

Similar to the proof of Theorem 1, we can prove the following theorem.

**Theorem 2** Let $H$ be an M-matrix and

$$-b_{i,j} < \beta_i < \frac{b_{i,j}}{1 - b_{i,j}} - b_{j,i}, -c_{j,i} < \delta_j < \frac{c_{j,i}}{1 - c_{j,i}} - c_{j,i},$$

for some $i \in \{2, 3, ..., p\}$, $j \in \{2, 3, ..., q\}$. Then $P_i' H$ is an M-matrix.

Now, we will show that in the case that the GMTS converges; the preconditioned GMTS methods converge faster.

**Theorem 3** Let $T$ and $T'_i$ be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively, assume that the matrix $H$ is irreducible, $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $0 \leq D_i \leq \hat{D}$, $0 \leq \tilde{D}_i \leq \hat{D}_i$, $0 \leq L_i \leq \hat{L}_i$, $0 \leq \tilde{L}_i \leq \hat{L}_i$, $b_{i,j} > 0$, $c_{j,i} > 0$, for some $i \in \{2, 3, ..., p\}$, $j \in \{2, 3, ..., q\}$.

$$-b_{j,i} < \alpha_i < \frac{b_{j,i}}{1 - b_{j,i}} - b_{j,i}, -c_{j,i} < \gamma_j < \frac{c_{j,i}}{1 - c_{j,i}} - c_{j,i},$$

for some $i \in \{2, 3, ..., p\}$, $j \in \{2, 3, ..., q - 1\}$.

If $\rho(T) < 1$, $\bar{D}_i \leq D_i$ and $\bar{L}_i \leq L_i$, then $\rho(T'_i) \leq \rho(T)$.

**Proof.** As the matrix $H$ is irreducible, so the $P_i' H$ is irreducible. So by Lemma 5, the matrix $T$ and $T'_i$ are irreducible. Consider the GMTS splitting for the matrix $H$, $H = M - N$, where

$$M = \hat{D} + D_i + L_i - \hat{L}, N = D_i + L_i + \hat{U}.$$  

Obviously, $H = M - N$ is a regular splitting and by the assumption $\rho(M^{-1}N) < 1$, we know that $H$ is an M-matrix. From Theorem 1, we can get that $P_i' H$ is also an M-matrix. Thus, from Lemma 7, we know that $P_i = \tilde{M}_i - \bar{N}_i$ is a regular splitting. Therefore, as $H$ is an M-matrix, we can get

$$\rho(T'_i) = \rho(\tilde{M}_i^{-1}\bar{N}_i) < 1.$$  

Now, we define the following splitting for the matrix $H$, $H = M_i' - N_i'$, in which $M_i' = (I + \bar{S}_i)^{-1}\tilde{M}_i$, $N_i' = (I + \bar{S}_i)^{-1}\bar{N}_i$ and

$$\bar{S}_i = \begin{pmatrix} S_i & 0 \\ 0 & V_i \end{pmatrix}.$$  

Consider the iteration matrix of the GMTS method $T = M^{-1}N_i$, it is easy to see that

$$M - \mathcal{M}_i = \begin{pmatrix} D_i - D_i' & 0 \\ L_{21} + L_{21} - L_i' & D_{22} - D_{22}' \end{pmatrix},$$

where

$$D_i = \begin{pmatrix} D_i' & 0 \\ 0 & D_{22}' \end{pmatrix} \leq \hat{D}, \quad \bar{D}_i = \begin{pmatrix} D_i' & 0 \\ 0 & D_{22}' \end{pmatrix} \leq \hat{D}_i,$$

$$L_i = \begin{pmatrix} 0 & 0 \\ 0 & L_{21} \end{pmatrix} \leq \hat{L}, \quad \bar{L}_i = \begin{pmatrix} 0 & 0 \\ 0 & L_{21} \end{pmatrix} \leq \hat{L}_i.$$  

It is known that $L_i' = (I + V_i)L$, hence $\bar{L}_i' - L = V_iL \leq 0$.

By computations, we know that $\bar{M}_i \leq M$, so $\bar{M}_i^{-1} \geq M^{-1}$.

Consequently,

$$M^{-1} \leq \bar{M}_i^{-1} \leq \bar{M}_i (I + S_i) = (M_i')^{-1}.$$  

From Lemma 4, we deduce that

$$\rho(\bar{M}_i^{-1}\bar{S}_i) = \rho((M_i')^{-1}S_i) \leq \rho(M^{-1}N_i),$$

so $\rho(\bar{T}_i') \leq \rho(T)$.

Similar to the proof of Theorem 3, we can prove the following theorem.

**Theorem 4** Let $T$ and $T'_i$ be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively, assume that the matrix $H$ is irreducible, $L \leq 0$, $U \leq 0$, $B \geq 0$, $C \geq 0$, $0 \leq D_i \leq \hat{D}$, $0 \leq \tilde{D}_i \leq \hat{D}_i$, $0 \leq L_i \leq \hat{L}_i$, $0 \leq \tilde{L}_i \leq \hat{L}_i$, $b_{i,j} > 0$, $c_{j,i} > 0$, for some $i \in \{2, 3, ..., p\}$, $j \in \{2, 3, ..., q\}$.

$$-b_{j,i} < \alpha_i < \frac{b_{j,i}}{1 - b_{j,i}} - b_{j,i}, -c_{j,i} < \gamma_j < \frac{c_{j,i}}{1 - c_{j,i}} - c_{j,i},$$

for some $i \in \{2, 3, ..., p\}$, $j \in \{2, 3, ..., q - 1\}$.

If $\rho(T) < 1$, $\bar{D}_i \leq D_i$ and $\bar{L}_i \leq L_i$, then $\rho(T'_i) \leq \rho(T)$.

IV. **Example**

Consider

$$H = \begin{pmatrix} I - B & U \\ L & I - C \end{pmatrix}.$$  

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where $B = \left( b_i \right)_{p \times p}$, $C = \left( c_{ij} \right)_{(n-p) \times (n-p)}$, $L = \left( l_{ij} \right)_{(n-p) \times p}$, and $U = \left( u_i \right)_{p \times (n-p)}$ with

$$b_i = \frac{1}{10 \times (i + 1)}, \quad i = 1, 2, \cdots, p,$$

$$b_j = \frac{1}{30} - \frac{1}{30 \times j + i}, \quad i < j, \quad i = 1, 2, \cdots, p - 1, \quad j = 2, \cdots, p,$$

$$b_j = \frac{1}{30} - \frac{1}{30 \times (i - j + 1) + i}, \quad i > j, \quad i = 2, \cdots, p, \quad j = 1, 2, \cdots, p - 1,$$

$$c_i = \frac{1}{10 \times (p + i + 1)}, \quad i = 1, 2, \cdots, n - p,$$

$$c_j = \frac{1}{30 \times (p + j) + p + i}, \quad i < j, \quad i = 1, 2, \cdots, n - p, \quad j = 1, 2, \cdots, n - p,$$

$$c_j = \frac{1}{30 \times (i - j + 1) + p + i}, \quad i > j, \quad i = 2, \cdots, n - p, \quad j = 1, 2, \cdots, n - p,$$

$$k_j = \frac{1}{30 \times (p + j) + i}, \quad i = 1, 2, \cdots, p, \quad j = 1, 2, \cdots, n - p.$$

In the experiments, the auxiliary are chosen such that

$$D_1 = 0.5(1 - \omega) l_1, \quad D_1 = 0.5(1 - \omega_1) l_1,$$

$$L_1 = 0.5(1 - \gamma_1) L_1, \quad L_1 = 0.5(1 - \gamma) L_1.$$

From Table I, we see that these results accord with Theorems 3-4.

### TABLE I

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#### REFERENCES


