Stability Analysis of Fractional Order Systems with Time Delay
Hong Li, Shou-Ming Zhong, Hou-Biao Li

Abstract—In this paper, we mainly study the stability of linear and interval linear fractional systems with time delay. By applying the characteristic equations, a necessary and sufficient stability condition is obtained firstly, and then some sufficient conditions are deserved. In addition, according to the equivalent relationship of fractional order systems with order \(0 < \alpha \leq 1\) and with order \(1 \leq \beta < 2\), one may get more relevant theorems. Finally, two examples are provided to demonstrate the effectiveness of our results.

Keywords—Fractional order systems, Time delay, Characteristic equation.

I. INTRODUCTION

RECENTLY, fractional order systems have considerable importance mainly due to the following two facts. First, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes, such as dielectric polarization [2] and electrolyte polarization [1], electrolyte polarization [2] and electromagnetic wave [3]. The advantages of the fractional order systems are that we have more degrees of freedom in the model and that a memory is included in the model. Second, fractional order controllers such as CRONE controller [4], TID controller [5] and fractional PID controller [6] have so far been implemented to enhance the robustness and the performance of the closed loop control system.

For commensurate fractional order systems, some powerful criteria have been proposed. The most well known one is Matignon’s stability theorem [7]. It permits us to check the stability system through the location in the complex plane of the dynamic matrix eigenvalues of the state space like system representation. Matignon’s theorem is the starting point of several results in the field. In addition, LMI approach ([8]-[10]) and Lyapunov approach ([11],[12]) are also used to investigate the stability of fractional order linear time invariant systems.

Recently, a finite time stability analysis of fractional time delay systems is firstly presented and reported on paper [13]. In addition, in [14], a stability test procedure is proposed for linear nonhomogeneous fractional order systems with a pure time delay by using a recently obtained generalized Gronwall’s inequality.

In this paper, we mainly research the stability of linear fractional systems with time delay. By using the characteristic equation for the system, we firstly give a necessary and sufficient stability condition, then, a sufficient condition is deserved. We also studied the stability of interval linear fractional system with time delay. Finally, two examples are provided to demonstrate the effectiveness of our results.

II. PROBLEM FORMULATION AND PRELIMINARIES

The differ-integral operator, denoted by \(\alpha D^\alpha_t\), is a combined differentiation and integration operator commonly used in fractional calculus which is defined by:

\[
\alpha D^\alpha t \begin{cases} 
\frac{d^\alpha}{dt^\alpha}, & \alpha > 0 \\
1, & \alpha = 0 \\
\int_0^t (\tau)^{-\alpha} d\tau, & \alpha < 0.
\end{cases}
\]

However, there exist some different definitions for fractional derivatives [15]. The most commonly used definitions are the Grünwald-Letnikov, Riemann-Liouville and Caputo ones. The Caputo definition is sometimes called smooth fractional derivative in literature because it is suitable to be treated by the Laplace transform technique, while the Riemann-Liouville definition is unsuitable.

In the rest of the paper, \(D^\alpha\) is used to denote the Caputo fractional derivative of order \(\alpha\)

\[
D^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - m)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau,
\]

where \(m\) is an integer satisfying \(m - 1 < \alpha \leq m\).

This paper focuses on the case that the fractional order is \(0 < \alpha \leq 1\).

Firstly, let us consider the linear fractional system with time delay described by the following form:

\[
\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bx(t-\tau)
\]

with the initial condition \(x(t_0 + \tau) = \psi(t) \in C[-\Delta, 0]\). Here \(0 < \alpha \leq 1\) is the fractional commensurate order, \(x(t) \in \mathbb{R}^n\) denotes the state vector, \(A\) and \(B \in \mathbb{R}^{n \times n}\) are the system matrices, and \(\tau > 0\) is pure time delay.

If the matrix \(A, B\) is uncertain, then the FO-LTI interval system can be described by the state space equation of the form

\[
\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) - Bx(t-\tau)
\]

where

\[
A \in [A^m, A^M] = \{a_{ij} : a_{ij} \leq a_{ij} \leq a_{ij}, 1 \leq i, j \leq n\}, \\
B \in [B^m, B^M] = \{b_{ij} : b_{ij} \leq b_{ij} \leq b_{ij}, 1 \leq i, j \leq n\}.
\]
If the fractional commensurate order $\alpha$ is uncertain, then the fractional-order interval system without delay can be described by the state space equation of the form
\[
d^\alpha x(t) = Ax(t)
\]
where $\alpha \in [\alpha_1, \alpha_2]$, $\alpha_1, \alpha_2 \in \mathbb{R}$.

To prove the main results in the next section, we need the following lemma.

**Lemma 1** ([16]). Let $R, T, V \in \mathbb{C}^{n \times n}$. If $|R| \leq v$, then
\[
|RT| \leq |R||T| \leq V|T|,
\]
\[
|R + T| \leq |R| + |T| \leq V + |T|,
\]
\[
\rho(R) \leq \rho(|R|) \leq \rho(V).
\]

**Proof.** Taking Laplace transform on both sides of system (2), we obtain
\[
s^\alpha X(t) \equiv \sum_{k=0}^{n-1} s^{\alpha-k-1} f^k(0) + AX(t) + Be^{-st} X(t) + Be^{-st} \int_0^t e^{-s(t-t')} x(t') dt' = 0
\]
where $X(t)$ is the Laplace transform of $x(t)$ with $X(0) = \mathcal{L}(x(t))$. Next, we rewrite (5) as follows:
\[
D(s)X(s) = b(s)
\]
in which
\[
D(s) = \text{det}(s^\alpha I - A - Be^{-st})
\]
\[
b(s) = \sum_{k=0}^{n-1} s^{\alpha-k-1} f^k(0) + Be^{-st} \int_0^t e^{-s(t-t')} x(t') dt'.
\]

Then, according to the paper [17], we can easily prove the theorem.

**Theorem 1.** If all the roots of the characteristic equation $D(s) = \text{det}(s^\alpha I - A - Be^{-st}) = 0$ have negative real parts, then the zero solution of system (2) is Lyapunov asymptotically stable.

**Proof.** Taking Laplace transform on both sides of system (2), we obtain
\[
s^\alpha X(t) = \sum_{k=0}^{n-1} s^{\alpha-k-1} f^k(0) + AX(t) + Be^{-st} X(t) + Be^{-st} \int_0^t e^{-s(t-t')} x(t') dt' = 0
\]
where $X(t)$ is the Laplace transform of $x(t)$ with $X(0) = \mathcal{L}(x(t))$. Next, we rewrite (5) as follows:
\[
D(s)X(s) = b(s)
\]
in which
\[
D(s) = \text{det}(s^\alpha I - A - Be^{-st})
\]
\[
b(s) = \sum_{k=0}^{n-1} s^{\alpha-k-1} f^k(0) + Be^{-st} \int_0^t e^{-s(t-t')} x(t') dt'.
\]

Then, according to the paper [17], we can easily prove the theorem.

**Theorem 2.** The system (2) is Lyapunov asymptotically stable, if the following inequalities are satisfied:

1. $\text{det}(s^\alpha I - A) \neq 0$, for $\Re(s) \geq 0$;
2. $\rho(F_M|B|) < 1$.

**Proof.** The characteristic equation of the system (2) is
\[
D(s) = \text{det}(s^\alpha I - A - Be^{-st}) = \text{det}(s^\alpha I - A)\text{det}(I - (s^\alpha I - A)^{-1} Be^{-st}).
\]

Therefore, if we can show that
\[
\text{det}(I - (s^\alpha I - A)^{-1} Be^{-st}) \neq 0, \quad \text{for } \Re(s) > 0
\]
then the system (2) is Lyapunov asymptotically stable. In fact, if we can show that $\rho((s^\alpha I - A)^{-1} Be^{-st}) < 1$, for $\Re(s) > 0$, (9) is satisfied by Lemma 1.

Now, using the Lemma 1, we obtain
\[
\rho((s^\alpha I - A)^{-1} Be^{-st}) \leq \rho(F_M|B|) < 1, \quad \text{for } \Re(s) > 0.
\]

Thus the proof is completed.

**Theorem 3.** The system (3) is Lyapunov asymptotically stable, if the following inequalities are satisfied:

1. $\text{det}(s^\alpha I - A) \neq 0$, for $\Re(s) \geq 0$;
2. $\rho(F_M|A_M + [B] + [M]) < 1$.

**Proof.** The characteristic equation of the system (3) is
\[
D(s) = \text{det}(s^\alpha I - (A + \Delta A) - (B + \Delta B)e^{-st}) = 0.
\]

Since $\text{det}(s^\alpha I - A) \neq 0$, for $\Re(s) \geq 0$ exists. We have
\[
D(s) = \text{det}(s^\alpha I - (A + \Delta A) - (B + \Delta B)e^{-st}) = \text{det}(s^\alpha I - A)\text{det}(I - (s^\alpha I - A)^{-1}(\Delta A + (B + \Delta B)e^{-st})).
\]

In the view of $|\arg \lambda(A)| > \frac{\pi}{2}$, $(s^\alpha I - A)^{-1}$ exists. Therefore, if we can show that, for $\Re(s) > 0$,
\[
\text{det}(I - (s^\alpha I - A)^{-1}(\Delta A + (B + \Delta B)e^{-st})) < 0,
\]

then the system (3) is Lyapunov asymptotically stable. In fact, if we can show that $\rho((s^\alpha I - A)^{-1}(\Delta A + (B + \Delta B)e^{-st})) < 1$, for $\Re(s) > 0$, the equation (13) is satisfied by Lemma 1.

Now, by Lemma 1, we may obtain
\[
\rho((s^\alpha I - A)^{-1}(\Delta A + (B + \Delta B)e^{-st})) \leq \rho(F_M|[\Delta A + (B + \Delta B)e^{-st}]) \leq \rho(F_M([A_M + [B] + [M]) < 1, \quad \text{for } \Re(s) > 0.
\]

This completes the proof.

In addition, according to the following equivalent theorem of fractional order systems with order $0 < \alpha \leq 1$ and with order $1 \leq \beta < 2$, one can obtain some other analogical conclusions on the order $1 \leq \beta < 2$ systems by the corresponding ones on the order $0 < \alpha \leq 1$ systems. Here we do not describe in detail.

**Theorem 4.** ([18]) All eigenvalues of the FO-LTI system (4) with order $0 < \alpha \leq 1$ and output $u(t) = 0$ settle in the unstable region if and only if the fractional-order system
\[
\frac{d^\beta x(t)}{dt^\beta} = -Ax(t), \quad 1 \leq \beta \leq 2 - \alpha < 2
\]
is asymptotically stable, see Fig. 1.
IV. NUMERICAL EXAMPLES

Example 1. Consider the stability of the following FO-LTI system with delay

\[
\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bx(t - \tau)
\]

where \( \alpha = \frac{1}{2} \), and

\[
A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}.
\]

Firstly, note that \( \det(s\alpha I - A) = s^{2\alpha} + 2 \), so \( \det(s\alpha I - A) \neq 0 \) for \( \Re(s) \geq 0 \).

\[
(s\alpha I - A)^{-1} = \begin{bmatrix} \frac{s^{\alpha}}{2s^{2\alpha} + 2} & -0.5 + \frac{s^{2\alpha}}{2s^{2\alpha} + 2} \\ \frac{s^{\alpha}}{2s^{2\alpha} + 2} & \frac{s^{2\alpha}}{2s^{2\alpha} + 2} \end{bmatrix}.
\]

so

\[
F_M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.
\]

Since the eigenvalues of matrix \( F_M \) are \( \lambda_1 = 0 \), \( \lambda_2 = 0.75 \). Therefore, from Theorem 2, we know that the fractional system (15) is Lyapunov stable.

Example 2. Consider the stability of the following interval FO-LTI system with delay

\[
\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bx(t - \tau)
\]

where \( \alpha = \frac{1}{2} \), and \( B \in [B_1, B_2] \)

\[
A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.1 & -0.2 \\ -0.5 & -0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}.
\]

Firstly, we have

\[
(s\alpha I - A)^{-1} = \begin{bmatrix} \frac{s^{\alpha}}{2s^{2\alpha} + 2} & -0.5 + \frac{s^{2\alpha}}{2s^{2\alpha} + 2} \\ \frac{s^{\alpha}}{2s^{2\alpha} + 2} & \frac{s^{2\alpha}}{2s^{2\alpha} + 2} \end{bmatrix}.
\]

so

\[
F_M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ -0.4 & 0.1 \end{bmatrix}.
\]

\[
\Delta A = 0, B_M = B_2 - B = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix},
\]

\[
F_M(|\bar{B}| + B_M) = \begin{bmatrix} 0.1 & 0.2 \\ 0.6 & 0.6 \end{bmatrix}.
\]

Since eigenvalues of the matrix \( F_M(|\bar{B}| + B_M) \) are \( \lambda_1 = 0.4 \), \( \lambda_2 = -0.1 \), so \( r(F_M(|\bar{B}| + B_M)) < 1 \). That is, from Theorem 3, the interval FO-LTI system (16) is Lyapunov stable.

V. CONCLUSIONS

In summary, this paper mainly presents a brief necessary and sufficient condition and some sufficient conditions for the stability of a class of FO-LTI system with uncertain parameters. The proposed method here is quite different from other ones in literature. Two simple examples also demonstrate that this method is feasible.

ACKNOWLEDGMENT

The authors sincerely thank reviewers for their valuable and detailed suggestions on the manuscript of this paper, which led to a substantial improvement. In addition, this paper was partly supported by the National Natural Science Foundation of China (11110171, 11271001, 11175443) and the Fundamental Research Funds for China Scholarship Council.

REFERENCES


Hong Li was born in 1979 in Sichuan, China. She received B.S. degree from China University of Geosciences (CUG), Wuhan, China, in 2002, and M.S. degrees in applied mathematics from university of electronic science and technology of China (UESTC), China, in 2006. She is currently as a lecturer in the school of mathematics sciences in UESTC and is pursuing a Ph.D degree. Her current research interests is mainly fractional-order systems. Email: sichuanhong@163.com.

Shou-ming Zhong was born in 1955 in Sichuan, China. He received B.S. degree in applied mathematics from UESTC, Chengdu, China, in 1982. From 1984 to 1986, he studied at the department of mathematics in Sun Yat-sen University, Guangzhou, China. From 2005 to 2006, he was a visiting research associate with the department of mathematics in university of Waterloo, Waterloo, Canada. He is currently as a full professor with school of mathematics sciences, UESTC. His current research interests include differential equations, neural networks, biomathematics and robust control. He has authored more than 80 papers in reputed journals such as the International Journal of Systems Science, Applied Mathematics and Computation, Chaos, Solitons and Fractals, Dynamics of Continuous, Discrete and Impulsive Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Acta Electronica Sinica, Control and Decision, and Journal of Engineering Mathematics etc.

Hou-biao Li received the M.Sc. and Ph.D. degrees in computational and applied mathematics from university of electronic science and technology of China (UESTC), China, in 2005 and 2007, respectively. He currently is an associate professor with the School of Mathematics Sciences, UESTC. His research interests involve numerical linear algebra, preconditioning technology and computational mathematics, etc. Email: lihoubiao0189@163.com