Nonlinear Equations with N-dimensional Telegraph Operator Iterated K-times

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Abstract—In this article, using distribution kernel, we study the nonlinear equations with N-dimensional telegraph operator iterated $k$-times.

Keywords—Telegraph operator, Elementary solution, Distribution kernel.

I. INTRODUCTION

The telegraph equation arises in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. The interaction of convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physics, chemistry and biology. Further, the telegraph equation is more suitable than ordinary diffusion in modeling reaction-diffusion for such branches of applied sciences. We refer the reader to [1]-[4] and the references therein.

Kamathii [5]-[6] has studied some properties and results of the distribution $e^{αx□δ}u(x)$ and solved the convolution equation

$$e^{αx□kδ}u(x) = e^{αx} \sum_{r=0}^{m} C_r □^r δ,$$

which is related to the ultra-hyperbolic equation, where $α = (α_1, α_2, \ldots, α_n)$, $αx = α_1x_1 + α_2x_2 + \cdots + α_nx_n$, $C_r$ are given constants for $r = 1, 2, \ldots, m$, $□^k$ is the $n$-dimensional ultra-hyperbolic operator iterated $k$ times defined by

$$□^k = \left( \frac{∂^2}{∂x_1^2} + \frac{∂^2}{∂x_2^2} + \cdots + \frac{∂^2}{∂x_p^2} - \frac{∂^2}{∂x_{p+1}^2} - \cdots - \frac{∂^2}{∂x_{p+q}^2} \right)^k$$

with $p + q = n$ and $δ$ is the Dirac-delta distribution with $□^k δ = δ, □^1 δ = □ δ$.

In this work, by applying the distribution $e^{αx□kδ}$, we study the elementary solution of the following $n$-dimensional telegraph equation

$$\left( \frac{∂^2}{∂x^2} + 2β \frac{∂}{∂t} + β^2 - ∆ \right)^k u(x,t) := T^k u(x,t) = δ(x,t),$$

where $∆$ is the $n$-dimensional Laplacian operator iterated $k$ times defined by

$$∆^k = \left( \frac{∂^2}{∂x_1^2} + \frac{∂^2}{∂x_2^2} + \cdots + \frac{∂^2}{∂x_n^2} \right)^k,$$

and $β$ is a positive constant. As an application, we solve the nonlinear equation with $n$-dimensional telegraph operator iterated $k$-times of the form

$$\left( \frac{∂^2}{∂x^2} + 2β \frac{∂}{∂t} + β^2 - ∆ \right)^k u(x,t) = f(x,t),$$

where $f(t, x)$ is a generalized function.

II. SOME DEFINITIONS AND LEMMAS

Definition 1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and write $v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2, p + q = n$.

Define by $Γ_+ = \{ x ∈ \mathbb{R}^n : x_1 > 0 \text{ and } v > 0 \}$ designating the interior of forward cone and $Γ_+$ designating its closure.

For any complex number $γ$, we define the function

$$R^H_γ(v) = \begin{cases} \frac{x^{(n-1)\frac{1}{2}}}{Γ(\frac{n-1}{2})} & \text{if } x ∈ Γ_+, \\ 0 & \text{if } x ∉ Γ_+ \end{cases}$$

(3)

where the constant $K_n(α)$ is given by the formula

$$K_n(γ) = \frac{Γ(\frac{n-1}{2})}{Γ(\frac{n}{2})} \frac{Γ(\frac{n+1}{2})}{Γ(\frac{n+2}{2})}$$

(4)

Let $supp R^H_γ(v) ⊂ Γ_+$ where $supp R^H_γ(v)$ denotes the support of $R^H_γ(v)$. The function $R^H_γ(v)$ is first introduced by Nozaki [7] and is called the ultra-hyperbolic kernel of Marcel Riesz. Moreover, $R^H_γ(v)$ is an ordinary function if Re($γ$) ≥ $n$ and is a distribution of $γ$ if Re($γ$) < $n$.

Definition 2. Let $x = (x_1, x_2, \ldots, x_n) ∈ \mathbb{R}^n$ and write $s = x_1^2 + x_2^2 + \cdots + x_n^2$.

For any complex number $β$, define the function

$$R^γ_β(s) = 2 - β - n - 2Γ \left( \frac{n - β}{2} \right)^s {β(β-n)\frac{1}{2}}$$

(5)

The function $R^γ_β(s)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if Re($β$) ≥ $n$ and is a distribution of $β$ if Re($β$) < $n$.

Lemma 1. [5] Let $L$ be the partial differential operator defined by

$$L = □ - 2 \left( \sum_{i=1}^{p} α_i \frac{∂}{∂x_i} - \sum_{j=p+1}^{p+q} α_j \frac{∂}{∂x_j} \right) + \sum_{i=1}^{p} α_i^2$$

(6)
Then
\[ (e^{\alpha x} \theta \delta) \ast u(x) = L^k u(x) = \delta \]
In addition, the unique elementary solution of (7) is given by
\[ u(x) = e^{\alpha x} R_{2k}^H(x), \]
where \( R_{2k}^H(x) \) is defined by (3) with \( \gamma = 2k \).

**Lemma 2.** \([8]\) \( e^{\alpha x} \theta(k) = (D - \alpha)^k \delta \) where \( D \equiv \frac{d}{dx} \) and \( e^{\alpha x} \theta(k) \) is a tempered distribution of order \( k \) with support 0.

**Lemma 3.** \([9]\) Let \( z \) be a complex number. Then
\[ \Gamma(z) \Gamma(z + \frac{1}{2}) = 2^{1-2z} \sqrt{\pi} \Gamma(2z), \quad z \neq 0, -1, -2, \ldots \]

### III. MAIN RESULTS

Now, we shall state and prove the following main results.

**Theorem 1.** Let \( T^k \) be the partial differential operator which iterated \( k \)-times defined by
\[ T^k = \left( \frac{\partial^2}{\partial t^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k, \]
where \( \Delta \) is the \( n \)-dimensional Laplacian operator and \( \beta \) is a given positive constant. Then \( u(x, t) = e^{-\beta t} M_{2k}(w) \) is a unique elementary solution of (1), where \( M_n(w) \) is defined by
\[ M_n(w) = \begin{cases} \frac{\omega^{(n-n/2)} \eta^{n+1}}{n!} & \text{if } t \in \Gamma_+, \\ 0 & \text{if } t \notin \Gamma_+, \end{cases} \]
where \( w = t^2 - x_1^2 - x_2^2 - \cdots - x_n^2 \) is the time and
\[ H_n+1(\eta) = \pi^{(n-1)/2} 2^{\eta-1} \Gamma \left( \frac{n+1}{2} \right) \Gamma(\eta/2). \]

**Proof.** Firstly, we define the \( n+1 \)-dimensional ultra-hyperbolic operator as
\[ \Box_{n+1} = \left( \frac{\partial^2}{\partial t^2} - \Delta \right). \]
Setting \( \alpha_2 = \alpha_3 = \cdots \alpha_n = 0 \), we have
\[ e^{\alpha_1 t} \theta(k) \] if \( t \in \Gamma_+, \]
\[ e^{\alpha_1 t} \theta(k) \] if \( t \notin \Gamma_+, \]
where \( \alpha_1 = \alpha_3 = \cdots \alpha_n = 0 \), and \( \alpha_2 = \alpha_4 = \cdots = \alpha_{n-2} = 0 \). We get
\[ e^{\alpha_1 t} \theta(k) \]
Using Lemma 2, we get
\[ e^{\alpha_1 t} \theta(k) \]
Substituting \( \alpha_1 = -\beta \), it follows that
\[ e^{-\beta t} \theta(k) \]
Convolving \( k \)-times for both sides of the above equation by \( e^{-\beta t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right) \delta(x, t) \), we have
\[ e^{-\beta t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right)^k \delta(x, t) = T^k \delta(x, t). \]

Then (1) can be written as
\[ T^k u(x, t) = e^{-\beta t} \left( \frac{\partial^2}{\partial t^2} - \Delta \right)^k \delta(x, t) \ast u(x, t) = \delta(x, t). \]

Convolving both sides of the above equation by \( e^{-\beta t} M_{2k}(w) \) and applying Lemma 1, we have
\[ u(x, t) = e^{-\beta t} M_{2k}(w), \]
where \( M_{2k}(w) \) is defined by (9) with \( \eta = 2k \).

**Theorem 2.** Given the equation
\[ \left( \frac{\partial^2}{\partial t^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k u(x, t) = f(x, t), \]
where \( f(x, t) \) is a given generalized function and \( u(x, t) \) is an unknown function. Then,
\[ u(x, t) = e^{-\beta t} M_{2k}(w) \ast f(x, t). \]

**Proof.** Convolving both sides of (11) by \( e^{-\beta t} M_{2k}(w) \) and applying the Theorem 1, we obtain (12) as required.

**Remark 3.** By using the method of proving Theorem 1 together with suitable modifications, we have \( u(x, t) = e^{-\beta t} (-1)^k R_{2k}^H(s) \) is a unique elementary solution of the following equation
\[ \left( \frac{\partial^2}{\partial t^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 + \Delta \right)^k u(x, t) = \delta(x, t), \]
where \( R_{2k}^H(s) \) is defined by Definition 2 with \( \beta = 2k \), \( s = t^2 + x_1^2 + x_2^2 + \cdots + x_n^2 \) and a constant \( n \) in (5) is replaced by \( n + 1 \).

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**REFERENCES**