Nonlinear Equations with N-dimensional Telegraph Operator Iterated K-times

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Abstract—In this article, using distribution kernel, we study the nonlinear equations with \( n \)-dimensional telegraph operator iterated \( k \)-times.

Keywords—Telegraph operator, Elementary solution, Distribution kernel.

I. INTRODUCTION

The telegraph equation arises in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. The interaction of convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physics, chemistry and biology. Further, the telegraph equation is more suitable than ordinary diffusion in modeling reaction-diffusion for such branches of applied sciences. We refer the reader to [1]-[4] and the references therein.

Kananthai [5]-[6] has studied some properties and results of the distribution \( e^{\alpha x} \Box^k \delta \) and solved the convolution equation

\[
e^{\alpha x} \Box^k \delta \ast u(x) = e^{\alpha x} \sum_{r=0}^{m} C_r \Box^r \delta,
\]

which is related to the ultra-hyperbolic equation, where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( \alpha x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \), \( C_r \) are given constants for \( r = 1, 2, \ldots, m \), \( \Box^k \) is the \( n \)-dimensional ultra-hyperbolic operator iterated \( k \) times defined by

\[
\Box^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k
\]

with \( p + q = n \) and \( \delta \) is the Dirac-delta distribution with \( \Box^0 \delta = \delta \), \( \Box^1 \delta = \Box \delta \).

In this work, by applying the distribution \( e^{\alpha x} \Box^k \delta \), we study the elementary solution of the following \( n \)-dimensional telegraph equation

\[
\left( \frac{\partial^2}{\partial x_1^2} + 2 \beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k u(x, t) := T^k u(x, t) = \delta(x, t),
\]

where \( \Delta \) is the \( n \)-dimensional Laplacian operator iterated \( k \) times defined by

\[
\Delta^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k
\]

and \( \beta \) is a positive constant. As an application, we solve the nonlinear equation with \( n \)-dimensional telegraph operator iterated \( k \)-times of the form

\[
\left( \frac{\partial^2}{\partial t^2} + 2 \beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k u(x, t) = f(x, t),
\]

where \( f(t, x) \) is a generalized function.

II. SOME DEFINITIONS AND LEMMAS

Definition 1. Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of \( \mathbb{R}^n \) and write

\[ v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \quad p + q = n. \]

Define by \( \Gamma \_+ = \{ x \in \mathbb{R}^n : x_1 > 0 \text{ and } \beta > 0 \} \) designating the interior of forward cone and \( \Gamma \_+ \) designating its closure.

For any complex number \( \gamma \), we define the function

\[
R^\gamma_H(v) = \begin{cases} \frac{(\gamma - n/2)}{K_n(\gamma)} & \text{if } x \in \Gamma \_+, \\ 0 & \text{if } x \notin \Gamma \_+, \end{cases}
\]

where the constant \( K_n(\gamma) \) is given by the formula

\[
K_n(\gamma) = \frac{\pi^{(n-1)/2} \Gamma \left( \frac{2n-2}{\gamma} \right) \Gamma \left( \frac{1-n}{2} \right) \Gamma \left( \frac{2}{\gamma} \right)}{\Gamma \left( \frac{2n+2}{\gamma} \right) \Gamma \left( \frac{2}{\gamma} \right)}.\]

Let \( \text{supp} R^\gamma_H(v) \subset \Gamma \_+ \) where \( \text{supp} R^\gamma_H(v) \) denotes the support of \( R^\gamma_H(v) \). The function \( R^\gamma_H(v) \) is first introduced by Nozaki [7] and is called the ultra-hyperbolic kernel of Marcel Riesz. Moreover, \( R^\gamma_H(v) \) is an ordinary function if \( \text{Re}(\gamma) \geq n \) and is a distribution of \( \gamma \) if \( \text{Re}(\gamma) < n \).

Definition 2. Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and write

\[ s = x_1^2 + x_2^2 + \cdots + x_p^2. \]

For any complex number \( \beta \), define the function

\[
R^\beta_H(s) = 2^{-\beta} \pi^{-n/2} \Gamma \left( \frac{n-\beta}{2} \right) \frac{s^{(\beta-n)/2}}{\Gamma \left( \frac{\beta}{2} \right)},
\]

The function \( R^\beta_H(s) \) is called the elliptic kernel of Marcel Riesz and is ordinary function if \( \text{Re}(\beta) \geq n \) and is a distribution of \( \beta \) if \( \text{Re}(\beta) < n \).

Lemma 1. [5] Let \( L \) be the partial differential operator defined by

\[
L = \Box - 2 \sum_{i=1}^{p} \alpha_i \frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q} \alpha_j \frac{\partial}{\partial x_j} + \left( \sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{p+q} \alpha_j^2 \right).\]
Then
\[ (e^{\alpha x} \square^k \delta) \ast u(x) = L^k u(x) = \delta \quad (7) \]
In addition, the unique elementary solution of (7) is given by
\[ u(x) = e^{\alpha x} R_{2k}^H(x), \]
where \( R_{2k}^H(x) \) is defined by (3) with \( \gamma = 2k \).

**Lemma 2.** [8] \( e^{\alpha t} \delta^{(k)}(x) = (D - \alpha)^k \delta \) where \( D \equiv \frac{\partial}{\partial x} \) and \( e^{\alpha t} \delta^{(k)}(x) \) is a tempered distribution of order \( k \) with support 0.

**Lemma 3.** [9] Let \( z \) be a complex number. Then
\[ \Gamma(z) = (z + \frac{1}{2}) = 2^{1-2z} \sqrt{\pi} \Gamma(2z), \quad z \neq 0, -1, -2, \ldots \quad (8) \]

### III. MAIN RESULTS

Now, we shall state and prove the following main results.

**Theorem 1.** Let \( T^k \) be the partial differential operator which iterated \( k \)-times defined by
\[ T^k = \left( \frac{\partial^2}{\partial x^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right)^k \quad (9) \]
where \( \Delta \) is the \( n \)-dimensional Laplacian operator and \( \beta \) is a given positive constant. Then \( u(x,t) = e^{-\beta t} M_{2k}(w) \) is a unique elementary solution of (1), where \( M_n(w) \) is defined by
\[ M_n(w) = \begin{cases} \frac{w^{n+1/3}}{n_{n+1/3}} & \text{if } t \in \Gamma^+, \\ 0 & \text{if } t \not\in \Gamma^+, \end{cases} \quad (10) \]
where \( w = t^2 - x_1^2 - x_2^2 - \cdots - x_n^2, t \) is the time and
\[ H_{n+1}(n) = \frac{n!(n-1)/2^{n-1}}{2^n} \Gamma \left( \frac{n+n+1}{2} \right) \Gamma \left( \frac{n}{2} \right). \quad (11) \]

**Proof.** Firstly, we define the \( n+1 \)-dimensional ultra-hyperbolic operator as
\[ \Box_{n+1} = \left( \frac{\partial^2}{\partial x^2} - \Delta \right). \]
Setting \( \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0 \), we have
\[ e^{\alpha(t,x) \square_{n+1}^k} \delta(x,t) = e^{\alpha t} \left( \frac{\partial^2}{\partial x^2} - \Delta \right)^k \delta(x,t) \quad (12) \]
Applying Lemma 3 for \( p = 1, q = n \) and \( p + q = n + 1 \), (3) and (4) are reduced to (9) and (10), respectively.

Indeed, we have \( \delta(x,t) = \delta(x) \delta(t) \) and \( e^{\alpha t} \delta(x) = \delta(x) \).

Using Lemma 2, we get
\[ e^{\alpha t} \left( \frac{\partial^2}{\partial x^2} - \Delta \right) \delta(x,t) = e^{\alpha t} \frac{\partial^2}{\partial x^2} \delta(x,t) - e^{\alpha t} \Delta \delta(x,t) \]
substituting \( \alpha_1 = -\beta \), it follows that
\[ e^{-\beta t} \left( \frac{\partial^2}{\partial x^2} - \Delta \right) \delta(x,t) = \left( \frac{\partial^2}{\partial x^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right) \delta(x,t) \]
\[ = T \delta(x,t) \quad (13) \]
Convolving \( k \)-times for both sides of the above equation by \( e^{-\beta t} \left( \frac{\partial^2}{\partial x^2} - \Delta \right) \delta(x,t) \), we have
\[ e^{-\beta t} \left( \frac{\partial^2}{\partial x^2} - \Delta \right) \delta(x,t) \ast \cdots \ast e^{-\beta t} \left( \frac{\partial^2}{\partial x^2} - \Delta \right) \delta(x,t) = e^{-\beta t} \left( \frac{\partial^2}{\partial x^2} - \Delta \right)^k \delta(x,t) \]
\[ = T \delta(x,t) \ast \cdots \ast T \delta(x,t) \]

Then (1) can be written as
\[ T^k u(x,t) = e^{-\beta t} \left( \frac{\partial^2}{\partial x^2} - \Delta \right)^k \delta(x,t) \ast u(x,t) = \delta(x,t) \quad (14) \]
Convolving both sides of the above equation by \( e^{-\beta M_{2k}(w)} \) and applying Lemma 1, we have
\[ u(x,t) = e^{-\beta M_{2k}(w)} \ast f(x,t) \quad (15) \]
where \( M_{2k}(w) \) is defined by (9) with \( \eta = 2k \).

**Theorem 2.** Given the equation
\[ \left( \frac{\partial^2}{\partial x^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right) u(x,t) = f(x,t), \quad (16) \]
where \( f(x,t) \) is a given generalized function and \( u(x,t) \) is an unknown function. Then,
\[ u(x,t) = e^{-\beta M_{2k}(w)} \ast f(x,t) \quad (17) \]
**Proof.** Convolving both sides of (16) by \( e^{-\beta M_{2k}(w)} \) and applying the Theorem 1, we obtain (12) as required.

**Remark 3.** By using the method of proving Theorem 1 together with suitable modifications, we have \( u(x,t) = e^{-\beta t} \left( -1 \right)^k R_{2k}^S(s) \) is a unique elementary solution of the following equation
\[ \left( \frac{\partial^2}{\partial x^2} + 2\beta \frac{\partial}{\partial t} + \beta^2 - \Delta \right) u(x,t) = \delta(x,t), \quad (18) \]
where \( R_{2k}^S(s) \) is defined by Definition 2 with \( \beta = 2k, s = t^2 + x_1^2 + x_2^2 + \cdots + x_n^2 \) and a constant \( n \) in (5) is replaced by \( n + 1 \).

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**REFERENCES**