Numerical Solution for Integro-Differential Equations by Using Quartic B-Spline Wavelet and Operational Matrices

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Abstract—In this paper, Semi-orthogonal B-spline scaling functions and wavelets and their dual functional are presented to approximate the solutions of integro-differential equations. The B-spline scaling functions and wavelets, their properties and the operational matrices of derivative for this function are presented to reduce the solution of integro-differential equations to the solution of algebraic equations. Here we compute B-spline scaling functions of degree 4 and their dual, then we will show that by using them we have better approximation results for the solution of integro-differential equations in comparison with less degrees of scaling functions.

Keywords—Integro-differential equations, Quartic B-spline wavelet, Operational matrices.

I. INTRODUCTION

The integral equation is a mathematical model of many evolutionary problems with memory arising from biology, chemistry, physics, engineering[3]. In recent years, many different basic functions have been used to estimate the solution of integral equations[9]-[21], such as orthogonal bases and wavelets. A differential equation can be replaced by an integral equation which incorporates its boundary conditions[1], as such each solution of the integral equation automatically satisfies these boundary conditions[3]. In application to discrete data sets, wavelets may be considered as basis functions generated by dilations and translations of a signal function[2]. In this paper, we develop a non-orthogonal (semi-orthogonal) wavelet using B-spline specially constructed for the bounded interval[17]-[16], this wavelet can be represented in a closed-form. Integro-Differential Equations(IDE) have applications in natural sciences and engineering. Recent work in the context of solution of these type of problems include spline approximation method[25], collocation method[12], wavelet basis method[4]-[5], hybrid Legendre polynomials and block-pulse functions approach[18]-[22], wavelet-Galerkin method[20], hybrid Taylor polynomials and block-pulse functions approach[19], Chebyshev collocation method[23] and Petrov-Galerkin method[30] etc.

Wavelet analysis has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis, time frequency analysis and fast algorithms for easy implementation[10]. Orthogonal wavelet either have infinite support or a nonsymmetric and in some cases, fractal nature. These properties can make them a poor choice for characterization of a function[11]-[29]. In contrast, the Semi-Orthogonal (SO) wavelets have finite support, both even and odd symmetry and simple analytical expressions, ideal attributes of a basis function[10]-[16].

In this present paper, we apply compactly supported quartic(five order) (SO) B-spline wavelets, specially constructed for the bounded interval [0, 1] to solve the first order Fredholm integro-differential equation of the form:

\[
\begin{align*}
\int_{0}^{1} g(x)y'(x) &= f(x) + \int_{0}^{1} k(x, t)y(t)dt \\
y(0) &= y_0
\end{align*}
\]

where \( f, g \) and \( k \) are given continuous functions and \( y \) is an unknown function to be determined.

The use of SO compactly supported spline wavelets is justified by their interesting properties[14]-[28]. These wavelets are smoothness, the larger are their supports in time(space). The order of vanishing moments usually increases with smoothness. Total positivity properties of splines have certain desirable properties from an approximation points of view[24].

The last decade demonstrates an augmentation of interest of B-spline wavelets. Solving integral and integro-differential equations using linear B-spline[15], quadratic B-spline[13] and cubic B-spline scaling functions[7]-[8] This wavelet family also found and application in image processing[2]. The efficiency of approximation using B-spline wavelets was compared with other families of orthogonal wavelets[26]. The obtained results indicate the effectiveness of high order B-spline in fault detection and localization.

II. B-SPLINE SCALING FUNCTIONS AND WAVELETS ON [0, 1]

We generate a doubly-indexed family of wavelets form \( \psi \) by dilating and translating: [11]

\[
\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a \neq 0, \quad a, b \in R
\]

The wavelets have been grouped in different families. These wavelets have an particularly desirable property that they are
zero in every where except of determined closed and bounded interval that we say so-called function has compact support. 
We can say generally that jth generation of daughters will have 2\(^j\) wavelets defined by:
\[
\psi_{j,k}(t) = \psi(2^j t - k), \quad 0 \leq k \leq 2^j - 1
\] (3)
The members of this generation will be constant on intervals of length 2\(^{-j+1}\). The first idea in studying of wavelets was this matter that we can write functions as linear combinations of the Father and Mother wavelets and first generation of daughters, This basis denote by \(B_j\). There is another basis of wavelets that is called sons wavelet. Here, we can define generations of sons wavelet by the following relation:
\[
\phi_{j,k}(t) = \psi(2^j t - k), \quad 0 \leq k \leq 2^j - 1
\] (4)
Now, assume \(S_j\) denote the set of 2\(^j\) functions \(\{\phi_{j,k}(t)\}_{k=0}^{2^j-1}\). Therefore, \(S_j\) will account as a basis for the inner product space \(V_j\). Vector space \(V_j\) with the basis \(S_j\). forms a nested sequence of subspaces \(V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots\) and using the basis \(B_j\) for \(V_j\), and orthogonal decomposition theorem we will have:
\[
V_j = V_{j-1} \oplus V_{j-1}^\perp = (V_{j-2} \oplus V_{j-2}^\perp) \oplus V_{j-1}^\perp = \ldots = V_0 \oplus V_0^\perp \oplus V_1^\perp \oplus \ldots V_{j-1}^\perp
\] (5)
The wavelets have especial particularities that all of them is gathered in a collection of Multi Resolution Analysis (MRA). Multi Resolution Analysis of \(L^2(R)\) is defined as a sequence of closed subspaces \(V_j\) of \(L^2(R)\), \(j \in Z\), with the following properties[11]:
1. \(V_j \subset V_{j+1}\)
2. \(f(x) \in V_j \iff f(2x) \in V_{j+1}\)
3. \(f(x) \in V_0 \iff f(x+1) \in V_0\)
4. \(\bigcup_{j} V_j\) is dense in \(L^2(R)\) and \(\bigcap_{j} V_j = \phi\)
5. A scaling function \(\phi \in V_0\) with a non vanishing integral, exists such that the collection \(\phi\) is Riesz basis of \(V_0\)

Some of the important properties relevant to the present work are given below:
1. vanishing moments: a wavelet is said to be having a vanishing moment of order \(m\) if
\[
\int_{-\infty}^{\infty} x^p \psi(x)dx = 0; \quad p = 0, \ldots, m - 1
\] (6)
All wavelets must satisfy the above condition for \(p = 0\).
2. Semi – orthogonality: the wavelets \(\psi_{j,k}\) form a semi-orthogonal basis if
\[
\langle \psi_{j,k}, \psi_{i,s} \rangle = 0; \quad i \neq j; \quad \forall i, j, k, s \in Z
\] (7)
The generalization to biorthogonal wavelets has been considered to gain more flexibility. Here, a dual scaling function \(\tilde{\phi}\) and a dual wavelet \(\tilde{\psi}\) exist that generate a Dual Multi Resolution Analysis (DMRA) with subspaces \(\tilde{V}_j\) and \(\tilde{W}_j\), such that
\[
\tilde{V}_j \perp W_j \quad \text{and} \quad V_j \perp \tilde{W}_j
\] (8)
and consequently
\[
\tilde{W}_j \perp W_{j'} \quad \text{for} \quad j \neq j'
\] (9)
When semi-orthogonal wavelets are constructed from B-spline of order \(m\), the lowest octave level \(j = j_0\) is determined in[10] by
\[
2^{j_0} \geq 2m - 1
\] (10)
so as to give a minimum of one complete wavelet on the interval [0,1]. In this paper we will use a wavelet generated by a quartic B-spline\((m = 5)\) cardinal B-spline function.
From(10), the five order B-spline lowest level, which must be an integer, is determined to \(j_0 = 4\). For each level \(j \geq j_0\) this constrains all octave levels to \(j \geq 4\).

A. Definition
Let \(m\) and \(n\) be two positive integers and
\[
a = x_{m+1} = \ldots = x_0 < x_1 < \ldots < x_n = x_{n+m-1} = b
\] (11)
be an equally spaced knots sequence. The functions
\[
B_{m,j}(x) = \frac{x-x_j}{x_{j+m-1}-x_j} B_{m-1,j}(x)
\]
\[
+ \frac{x_{j+m}-x}{x_{j+m}-x_{j+1}} B_{m-1,j+1}(x)
\] (12)
are called cardinal B-spline functions of order \(m \geq 2\) for the knot sequence \(X = \{x_k\}_{k=0}^{2m-1}\).

For the sake of simplicity, suppose \([a,b] = [0,n]\) and \(x_j = j, j = 0, \ldots, n\). The \(B_{m,j}(x) = B_m(x-k), j = 0, \ldots, n - m\), are interior B-spline functions, while the remaining \(B_{m,j,X}(x) = -m + 1, \ldots, -1\) and \(j = n - m + 1, \ldots, n - 1\) are boundary B-spline functions[14], for the bounded interval [0,n]. Since the boundary B-spline functions at 0 are symmetric reflections of those at \(n\), it is sufficient to construct only the first half functions by simply replacing \(x\) with \(n - x\).
By considering the interval \([a,b] = [0,1]\), at any level \(j \in Z^+\), the discrete step is \(2^{-j}\), and this generates \(n = 2^j\) number of segments in [0,1] with knot sequence[31]
\[
X^j = \{ x_{1-m}^j = x_0^j, \ldots, x_0^j = 0, x_k^j = k 2^{-j} \quad k = 1, \ldots, 2^j - 1 \}
\]
\[
x_{2^j}^j = x_{2^j-1}^j = \ldots = x_{2^j + m - 1}^j = 0
\] (14)
For each level \(j \geq j_0\) the scaling function of order \(m\) can be defined as allows:
The B-spline wavelet can be defined recursively by the convolution:

\[
\psi_m(x) = \int_{-\infty}^{\infty} \varphi_{m-1}(x-t) \varphi_1(t) dt = \int_0^1 \varphi_{m-1}(x-t) dt
\]  

where \( \varphi_1(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{else} \end{cases} \)  

The construction of the scaling function of \( m \)-th order B-spline wavelet is based on the two scale relation:

\[
\varphi_m(x) = \sum_{k=0}^{m} p_k \varphi_m(2x - k)
\]  

where \( p_k \) is the two scale coefficient and can be expressed as a combination:

\[
p_k = 2^{1-m} \left( \frac{m}{k} \right), 0 \leq k < m
\]  

The two-scale relation for \( m \)-th order B-spline wavelets is given by:

\[
\psi_m = \sum_{k=0}^{3m-2} q_k \varphi_m(2x - k)
\]

\[
q_k = (-1)^k 2^{1-m} \sum_{l=0}^{m} \left( \frac{m}{l} \right) \varphi_{2m}(k - l + 1)
\]

The decomposition relation for \( m \)-th order B-spline wavelet is:

\[
\varphi_m(2x - l) = \sum_k (a_{l-2k} \varphi(x - k) + b_{l-2k} \psi(x - k)), l \in Z
\]

where decomposition sequences \( \{a_k\} \) and \( \{b_k\} \) are as follows:

\[
a_k = \frac{(-1)^{k+1}}{2} \sum_l q_{-k+2m-2l-1} c_{l,2m}
\]

\[
b_k = \frac{(-1)^{k+1}}{2} \sum_l p_{-k+2m-2l-1} c_{l,2m}
\]

In (27) and (28) the coefficients sequence \( \{c_{k,m}\} \) is presented by \( m \)-th order fundamental cardinal spline functions:

\[
L_m(x) = \sum_{k=-\infty}^{\infty} c_{k,m} \varphi_m \left( \frac{m}{2} + x - k \right)
\]

To obtain the coefficient sequences, using an analytical relation for B-spline wavelets with order \( m < 3 \). For higher values of \( m \) obtaining the analytical solutions become very difficult, and for values of \( m \) greater than 5, it is impossible in the light of Abel-Ruffini theorem. Therefore, the analytical formula was omitted here. Another way of obtaining the coefficient sequences is to form the bi-infinite system of equations as follows:

\[
\sum_{k=-\infty}^{\infty} c_{k,m} \varphi_m \left( \frac{m}{2} + j - k \right) = \delta_{j,0}, j \in Z
\]

The coefficients sequence \( \{c_{k,m}\} \) is infinite for \( m \geq 3 \), so that (29) does not vanish identically outside any compact set. However, these coefficients decay to zero exponentially fast as \( k \to \infty \), which implies decaying to zero of (29) as \( x \to \pm \infty \).

IV. QUARTIC B-SPLINE WAVELET (\( m = 5 \))

Quartic B-spline \( \varphi_5(x) \) scaling function is given by the next recursive relation:

\[
\varphi_m(x) = \sum_{k=0}^{m} p_k \varphi_m(2x - k)
\]
relations for $p$ and $q$  

\begin{align*}
(x_j - k)^4 = & \frac{5(x_j - k)^2}{6} - \frac{5}{6} x_j - k + 1 \\
& + \frac{5(x_j - k)^2}{4} - \frac{5}{4} x_j - k + 2 \\
& + \frac{5(x_j - k)^2}{4} - \frac{5}{4} x_j - k + 3 \\
& + \frac{5(x_j - k)^2}{4} - \frac{5}{4} x_j - k + 4 \\
& + \frac{5(x_j - k)^2}{4} - \frac{5}{4} x_j - k + 5 \\
\end{align*}

\( k \leq x_j < k + 1 \)

\( k + 1 \leq x_j < k + 2 \)

\( k + 2 \leq x_j < k + 3 \)

\( k + 3 \leq x_j < k + 4 \)

\( k + 4 \leq x_j < k + 5 \)

\textit{else} \hspace{1cm} (34)

So the corresponding scaling function is:

\[
\varphi_{j,k}(x) = 
\begin{cases}
-\frac{x^6}{6} + \frac{5x^4}{6} - \frac{5x^2}{2} + \frac{5x^2}{2} - \frac{x}{6} & 0 \leq x < 1 \\
\frac{x^4}{2} - \frac{5x^2}{2} + \frac{5x^2}{2} + \frac{x}{6} + \frac{1}{6} & 1 \leq x < 2 \\
\frac{x^4}{2} - \frac{5x^2}{2} + \frac{5x^2}{2} + \frac{x}{6} + \frac{1}{6} & 2 \leq x < 3 \\
\frac{x^4}{2} - \frac{5x^2}{2} + \frac{5x^2}{2} + \frac{x}{6} + \frac{1}{6} & 3 \leq x < 4 \\
\frac{x^4}{2} - \frac{5x^2}{2} + \frac{5x^2}{2} + \frac{x}{6} + \frac{1}{6} & 4 \leq x < 5 \\
0 & \text{else} \\
\end{cases}
\]

where the compact support in the range \([0, m]\) referring to the property B-spline scaling functions. Two scale sequences \(\{p_k\}_{k=0}^5\) and \(\{q_k\}_{k=0}^{13}\) are as follow. Based on them two scale relations for \(\varphi_5(x)\) and \(\psi_5(x)\) can be constructed using (22) and (24) respectively.

\[
\{p_k\}_{k=0}^5 = \left\{ \frac{1}{16}, \frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{1}{16} \right\}
\]

\[
\{q_k\}_{k=0}^{13} = \left\{ \frac{1}{5806080}, \frac{169}{1935360}, \frac{2141}{725760}, \frac{5197}{725760}, \frac{181440}{1935360}, \frac{54289}{74339}, \frac{149693}{74339}, \frac{2141}{725760}, \frac{5197}{725760}, \frac{181440}{1935360}, \frac{54289}{74339}, \frac{149693}{74339}, \frac{169}{1935360} \right\}
\]

Figs. 1 and 2 show the scaling and wavelet function for quartic B-spline wavelet.

With the respective left and right hand side boundary scaling function. The actual coordinate position \(x\) is related to \(x_j\) according to \(x_j = 2^j x\).

\[
\varphi_5(2x - k) = 
\begin{cases}
\frac{(2x - k)^4}{24} - \frac{5(2x - k)^2}{6} + \frac{5(2x - k)^2}{4} - \frac{5}{4} x_j - k + 1 & k/2 \leq x < k/2 + 1/2 \\
\frac{(2x - k)^4}{24} - \frac{5(2x - k)^2}{6} + \frac{5(2x - k)^2}{4} - \frac{5}{4} x_j - k + 1/2 & k/2 + 1/2 \leq x < k/2 + 1 \\
\frac{(2x - k)^4}{24} - \frac{5(2x - k)^2}{6} + \frac{5(2x - k)^2}{4} - \frac{5}{4} x_j - k + 3/2 & k/2 + 1 \leq x < k/2 + 1 \\
\frac{(2x - k)^4}{24} - \frac{5(2x - k)^2}{6} + \frac{5(2x - k)^2}{4} - \frac{5}{4} x_j - k + 2 & k/2 + 2 \leq x < k/2 + 1 \\
0 & \text{else} \\
\end{cases}
\]

(35)

\[
\varphi(x) = \frac{1}{16} \varphi(2x) + \frac{5}{16} \varphi(2x - 1) + \frac{5}{8} \varphi(2x - 2) + \frac{5}{8} \varphi(2x - 3) + \frac{5}{16} \varphi(2x - 4) + \frac{1}{16} \varphi(2x - 5)
\]

(36)

\[
\varphi_{5,0}(x) = 
\begin{cases}
\frac{5(16x + 3)^3}{24} - \frac{5(16x + 3)^3}{6} + \frac{25(16x + 3)^3}{4} & 0 \leq x < 1/16 \\
0 & \text{else} \\
\end{cases}
\]

(37)

\[
\varphi_{5,1}(x) = 
\begin{cases}
\frac{5(16x + 3)^3}{24} - \frac{5(16x + 3)^3}{6} + \frac{25(16x + 3)^3}{4} & 0 \leq x < 1/16 \\
0 & \text{else} \\
\end{cases}
\]

(38)

\[
\varphi_{5,2}(x) = 
\begin{cases}
\frac{5(16x + 2)^3}{24} - \frac{5(16x + 2)^3}{6} + \frac{35(16x + 2)^3}{4} & 0 \leq x < 1/16 \\
\frac{5(16x + 2)^3}{24} - \frac{5(16x + 2)^3}{6} + \frac{35(16x + 2)^3}{4} & 0 \leq x < 1/16 \\
\frac{5(16x + 2)^3}{24} - \frac{5(16x + 2)^3}{6} + \frac{35(16x + 2)^3}{4} & 1/16 \leq x < 1/8 \\
0 & \text{else} \\
\end{cases}
\]

(39)
\( \varphi_5(-1)(x) = \begin{cases} 
\frac{-1}{2} \frac{16x}{16} + \frac{5(16x+1)^3}{4} + 3(16x+1)^2 + 
\frac{5(16x+1)^2}{4} & 0 \leq x < 1/16 \\
\frac{1}{2} \frac{16x}{16} + \frac{5(16x+1)^3}{4} + 3(16x+1)^2 + 
\frac{5(16x+1)^2}{4} & 1/16 \leq x < 1/8 \\
\frac{1}{2} \frac{16x}{16} + \frac{5(16x+1)^3}{4} + 3(16x+1)^2 + 
\frac{5(16x+1)^2}{4} & 1/8 \leq x < 3/16 \\
0 & 3/16 \leq x < 1/4 \\
else & 
\end{cases} \) (40)

\( \varphi_5(12)(x) = \begin{cases} 
\frac{1}{6} \frac{16x}{16} + \frac{5(16x-12)^3}{4} + 5(16x-12)^2 + 
\frac{5(16x-12)^2}{4} & 15/16 \leq x < 1 \\
\frac{1}{6} \frac{16x}{16} + \frac{5(16x-12)^3}{4} + 5(16x-12)^2 + 
\frac{5(16x-12)^2}{4} & 7/8 \leq x < 15/16 \\
\frac{1}{6} \frac{16x}{16} + \frac{5(16x-12)^3}{4} + 5(16x-12)^2 + 
\frac{5(16x-12)^2}{4} & 13/16 \leq x < 7/8 \\
0 & 12/16 \leq x < 13/16 \\
else & 
\end{cases} \) (41)

\( \varphi_5(13)(x) = \begin{cases} 
\frac{-x}{6} + \frac{(16x-13)^3}{4} + 3(16x-13)^2 + 
\frac{5(16x-13)^2}{4} & 15/16 \leq x < 1 \\
\frac{-x}{6} + \frac{(16x-13)^3}{4} + 3(16x-13)^2 + 
\frac{5(16x-13)^2}{4} & 7/8 \leq x < 15/16 \\
\frac{-x}{6} + \frac{(16x-13)^3}{4} + 3(16x-13)^2 + 
\frac{5(16x-13)^2}{4} & 13/16 \leq x < 7/8 \\
0 & else 
\end{cases} \) (42)

\( \varphi_5(14)(x) = \begin{cases} 
\frac{-x}{6} + \frac{(16x-14)^3}{4} + 3(16x-14)^2 + 
\frac{5(16x-14)^2}{4} & 15/16 \leq x < 1 \\
\frac{-x}{6} + \frac{(16x-14)^3}{4} + 3(16x-14)^2 + 
\frac{5(16x-14)^2}{4} & 7/8 \leq x < 15/16 \\
\frac{-x}{6} + \frac{(16x-14)^3}{4} + 3(16x-14)^2 + 
\frac{5(16x-14)^2}{4} & 13/16 \leq x < 7/8 \\
0 & else 
\end{cases} \) (43)

V. Function approximating using scaling function

For any positive integer \( M = j_0 \), a function \( f(x) \) defined over \([0, 1] \) may be represented by B-spline scaling functions as:

\[ f(x) = \sum_{k=4}^{2M-1} s_k \varphi_{M,k}(x) = S^T \Phi_M \] (46)

where

\[ S = [s_{-4}, s_{-3}, ..., s_{2M-1}] \]

\[ \Phi_M = [\Phi_{M,-4}, \Phi_{M,-3}, ..., \Phi_{M,2M-1}] \] (47)

with

\[ \psi(x) = \sum_{k=0}^{13} q_k \varphi(2x - k) \] (45)

\[ s_k = \int_{0}^{1} f(x) \varphi_{M,k}(x) dx, \ k = -4, -3, ..., 2M - 1 \] (48)
where \( \tilde{\varphi}_{M,k}(x) \) are dual functions of \( \varphi_{M,k}(x) \).

\[
\tilde{\Phi} = T_\Phi \Phi, \quad T_\Phi = (P_M)^{-1} \tag{49}
\]

These can be obtained by linear combinations of \( \varphi_{M,k}(x), k = -4, -3, ..., 2M - 1 \), as follows. Let \( \Phi_M \) be the dual functions of \( \Phi_M \) given by:

\[
\Phi_M = [\tilde{\varphi}_{M,-4}, \tilde{\varphi}_{M,-3}, ..., \tilde{\varphi}_{M,2M-1}] \tag{50}
\]

Using (47) and (48) we get:

\[
\int_{0}^{1} \phi_{M,j}(x) \phi_{M,j}(x) dx = I_1 \tag{51}
\]

where \( I_1 \) is \((2M + 4) \times (2M + 4)\) identity matrix. Let:

\[
P_M = \int_{0}^{1} \Phi_M \Phi_M^T dx \tag{52}
\]

The entry \((P_M)_{i,j}\) of the matrix \(P_M\) is calculated from:

\[
\int_{0}^{1} \varphi_{M,i}(x) \varphi_{M,j}(x) dx \tag{53}
\]

From (51) and (52) we get:

\[
\Phi_M = (P_M)^{-1} \Phi_M \tag{54}
\]

Furthermore, a function \( f(x) \) defined over \([0, 1]\) may be represented by B-spline wavelets as:

\[
f(x) = \sum_{k=-4}^{2^j-1} c_{m,k} \varphi_{5,k}(x) + \sum_{j=5}^{\infty} \sum_{k=-4}^{2^j-5} d_{j,k} \psi_{j,k}(x) \tag{55}
\]

If the infinite series in (55) is truncated at \( M \), then (55) can be written as:

\[
f(x) \simeq \sum_{k=-4}^{15} c_{m,k} \varphi_{5,k}(x) + \sum_{j=5}^{2^j-5} \sum_{k=-4}^{2^j-5} d_{j,k} \psi_{j,k}(x) = C^T \Psi(x) \tag{56}
\]

where \( \varphi_{5,k} \) and \( \psi_{j,k} \) are scaling and wavelets functions, respectively \( C \) and \( \Psi \) are \((2M + 4) \times 1\) vectors given by:

\[
C = [c_{-4}, c_{-3}, ..., c_{15}, d_{5,-4}, d_{5,-3},
\ldots, d_{5,7}, \ldots, d_{M,-M+1}, \ldots, d_{2M-5,-5}] \tag{57}
\]

\[
\Psi = [\varphi_{5,-4}, \varphi_{5,-3}, ..., \varphi_{5,15}, \psi_{5,-4}, \psi_{5,-3}, ..., \psi_{5,7}, ..., \psi_{M,-M+1}, ..., \psi_{M,2M-5}] \tag{58}
\]

with

\[
c_k = \int_{0}^{1} f(x) \tilde{\varphi}_{5,k}(x) dx, \quad k = -4, -3, ..., 15 \tag{59}
\]

\[
d_{j,k} = \int_{0}^{1} f(x) \tilde{\psi}_{j,k}(x) dx, \quad j = 5, 4, ..., M, \quad k = -4, -3, ..., 2^j - 5 \tag{60}
\]

where \( \tilde{\varphi}_{5,k}(x) \) and \( \tilde{\psi}_{j,k}(x) \) are dual functions of \( \varphi_{5,k}(x) \) and \( \psi_{j,k}(x) \) respectively. These can be obtained by linear combinations as follows:

\[
\Phi = [\varphi_{5,-4}(x), \varphi_{5,-3}(x), ..., \varphi_{5,15}(x)]^T \tag{61}
\]

\[
\Psi = [\psi_{5,-4}(x), \psi_{5,-3}(x), ..., \psi_{M,2M-5}(x)]^T \tag{62}
\]

VI. THE OPERATIONAL MATRICES OF DERIVATIVE

The differentiation of vectors \( \Phi_M \) in (47) can be expressed as:

\[
\Phi'_M = D_\Phi \Phi_M \tag{63}
\]

where \( D_\Phi \) is \((2M + 4) \times (2M + 4)\) operational matrices of derivative for B-spline scaling functions.

\[
D_\Phi = \int_{0}^{1} \Phi'_M(t) \Phi_M^T(t) dt
\]

\[
= (\int_{0}^{1} \Phi'_M(t) \Phi_M^T(t) dt)(P_M)^{-1} = F(P_M)^{-1} \tag{64}
\]

where

\[
F = \int_{0}^{1} \Phi'_M(t) \Phi_M^T(t) dt \tag{65}
\]

\( F \) is matrices \((2M + 4) \times (2M + 4)\) as follows:

\[
\begin{bmatrix}
\int_{0}^{1} \varphi'_{M,-4}(t) \varphi_{M,-4}(t) dt & \ldots & \int_{0}^{1} \varphi'_{M,-4}(t) \varphi_{M,2M-5}(t) dt \\
\vdots & \ddots & \vdots \\
\int_{0}^{1} \varphi'_{M,2M-5}(t) \varphi_{M,-4}(t) dt & \ldots & \int_{0}^{1} \varphi'_{M,2M-5}(t) \varphi_{M,2M-5}(t) dt
\end{bmatrix} \tag{66}
\]

Since the element \( \varphi_{M,k} \) in the vector \( \Phi_M \) given in (47) is nonzero between \( \frac{k}{2^j} \) and \( \frac{k+1}{2^j} \) for any entries of \( F_{j,k} \) we have:

\[
F_{j,k} = \int_{0}^{1} \varphi'_{M,j}(t) \varphi_{M,k}(t) dt = \int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} \varphi'_{M,j}(t) \varphi_{M,k}(t) dt \tag{67}
\]
VII. THE OPERATIONAL MATRICES OF DERIVATIVE USING WAVELETS

The differentiation of vectors Ψ in (58) can be expressed as:

\[ \Psi' = D_\Psi \Psi \]  

where \( D_\Psi \) is \((2^M + 4) \times (2^M + 4)\) operational matrices of derivative for B-spline wavelets. The matrix \( D_\Psi \) can be obtained by considering:

\[ \Psi = H \Phi_M \]  

where \( H \) is a \((2^M + 4) \times (2^M + 4)\) matrix, which can be calculated as follows:

\[ \Phi_j = [\psi_{j-4}, \psi_{j-3}, ..., \psi_{j-2^M}]^T \]

\[ \Psi_j = [\psi_{j-4}, \psi_{j-3}, ..., \psi_{j-2^M}]^T \]  

Using (32) and (70) we get

\[ \Phi_j = \alpha_j \Phi_{j+1} \]  

where \( \alpha_j, j = 5, ..., \) is \((2j + 4) \times (2j + 1 + 4)\) matrix. From (33) and (70) we have:

\[ \Psi_j = U_j \Phi_j + 1 \]  

where \( U_j, j = 4, ..., \) is \((2j') \times (2j + 1 + 4)\) matrix. Using (62), (71) and (72) matrix \( H \) obtain as follows:

\[
\begin{bmatrix}
\alpha_5 \times \alpha_4 \times \cdots \times \alpha_M \\
- \cdots - \\
U_5 \times \alpha_4 \times \cdots \times \alpha_M \\
- \cdots - \\
\vdots \\
U_{M-2} \times \alpha_{M-1} \times \alpha_M \\
- \cdots - \\
U_{M-1} \times \alpha_M \\
- \cdots - \\
U_M
\end{bmatrix}
\]  

From (64), (65) and (73) we get:

\[ \Psi' = H \Phi_M = H D_\Psi \Phi_M = HF(P_M)^{-1} \Phi_M = D_\Psi \Psi \]  

VIII. FIRST ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

In this section we solve first order Fredholm integro-differential equations of the form (1) by using B-spline wavelets. Let:

\[ w(x) = \int_0^1 k(x, t)y(t)dt \]  

For this purpose, we first approximate \( y(x) \) and \( w(x) \) to expand (68) as:

\[ y(x) = C^T \Psi(x) \]

\[ w(x) = \int_0^1 k(x, t)C^T \Psi(t)dt \]  

where \( \Psi(x) \) is defined in (58) and \( C \) is \((2^M + 4) \times 1\) unknown vector defined similarly to \( C \) in (57). We can approximate (77) using quadrature Newton-Cotes integration techniques as:

\[ w(x) = \int_0^1 k(x, t)C^T \Psi(t)dt = \sum_{i=1}^{n} \omega_x k(x, t_i) C^T \Psi(t_i) \]  

where \( \omega \) and \( t_i \) are weight and nodes of Newton-Cotes integration method. For approximate \( y'(x) \) to expand (68) as:

\[ y'(x) = C^T \Psi'(x) = C^T D_\Psi \Psi(x) \]  

From (1), (77) and (78), we get:

\[ g(x) C^T D_\Psi \Psi(x) = f(x) + w(x) \]  

Also using boundary values in (1) and (76) we have:

\[ C^T \Psi(0) = y_0 \]  

To find the solution \( y(x) \) in (76) we first collocate (80) in \( x_i = (2i - 1)/(2^M + 2) - 1, \) \( i = 1, ..., 2^M + 1 \), the resulting equation generates \( 2^M + 1 \) algebraic equations. The total unknowns for vector \( C \) in (76) is \( 2^{M+1} \). These can be obtained by using (80), (81).

IX. ERROR CONSIDERATION

In this section, we found an error bound for the presented method.

A. Theorem 1

Assume that \( f \in C^5[0, 1] \) is represented by quartic B-spline wavelets as (56), where has 5 vanishing moments, then [10]

\[ |d_j, k| \leq \mu \varepsilon \frac{2^{-6j}}{5} \]

where \( \mu = \max |f^{(5)}(t)|_{t \in [0, 1]} \) and \( \varepsilon = \int_0^1 |x^5 \psi_5(x)| dx \).

B. Theorem 2

Consider the previous theorem assume that \( e_j(x) \) be error of approximation in space \( V_j \), then

\[ |e_j(x)| = O(2^{-5j}) \]  

Thus, order of error depend on the level \( j \). Obviously, for larger level of \( j \), the error of approximation will be smaller.
We applied the method presented and solved equation. The exact solution of this problem is \( y(x) = e^x \). The absolute error for \( M = 4, 5 \) are shown in Table I.

3) Example 3:

\[
\begin{align*}
y'(x) = \frac{1}{1+x} - \frac{x}{2} - \ln(x+1) + \frac{1}{(\ln 2)^2} \int_0^1 \frac{y(t)}{t+1} dt + y(x) \\
y(0) = 0
\end{align*}
\]

The exact solution of this problem is \( y(x) = \ln(1 + x) \). The absolute error for \( M = 4, 5 \) are shown in Table III.
XI. CONCLUSION
In the present work, a technique has been developed for solving first-order Fredholm integro-differential equations. As the B-spline wavelets have the property of semi-orthogonality on the interval [0, 1], we can combine various bases with quartic B-spline wavelets to produce Hybrid functions, with the property of semi-orthogonality. The same approach can be used to solve other problems. The operational matrices of derivative for B-spline scaling functions and wavelets are given. The problem has been reduced to solving a system of algebraic equations and applications are demonstrated through numerical examples. The numerical examples show that the accuracy improves with increasing the M in quartic B-spline wavelets, and thus for better results, using the larger M is recommended.

REFERENCES

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