Quasi-Permutation Representations for the Group $SL(2, q)$ when Extended by a certain Group of order Two

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Abstract—A square matrix over the complex field with non-negative integral trace is called a quasi-permutation matrix. For a finite group $G$ the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rationals and the complex numbers are denoted by $q(G)$ and $c(G)$ respectively. Finally $r(G)$ denotes the minimal degree of a faithful rational valued complex character of $G$. The purpose of this paper is to calculate $c(G)$, $q(G)$ and $r(G)$ for the group $SL(2, q)$ when extended by a certain group of order two.

AMS Subject Classification : 20C15.

Keywords and phrases : General linear group, Quasi-permutation.

In [10] Wong defined a quasi-permutation group of degree $n$ to be a finite group $G$ of automorphisms of an $n$-dimensional complex vector space such that every element of $G$ has non-negative integral trace. Also Wong studied the extent to which some facts about permutation groups generalize to the quasi-permutation group situation. In [3] the authors investigated further the analogy between permutation groups and quasi-permutation groups. They also worked over the rational field and found some interesting results.

By a quasi-permutation matrix we mean a square matrix over the complex field $C$ with non-negative integral trace. For a given finite group $G$, let $q(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational field $Q$ and let $c(G)$ be the minimal degree of a faithful representation of $G$ by complex quasi-permutation matrices and finally let $r(G)$ denote the minimal degree of a faithful rational valued character of $G$. In this paper we will apply the algorithms in [1] to the group $K_3^2(2^n)$, where

$$K_3^2(n) = \langle SL(n, q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} \rangle .$$

We will prove

Theorem 1: A) Let $G = K_3^2(2)$ then $r(G) = 2$, $c(G) = q(G) = 4$.
B) Let $K_3^2(q)$, where $q = 2^n$. Then
1) If $q \equiv -1 \text{ mod } 3$ then $r(G) = q - 1$, $c(G) = q(G) = (q - 1)$
2) Otherwise : $r(G) = q$, $c(G) = q(G) = 2q$

Let $SL(n, q)$ denote the special general linear group of a vector space of dimension $n$ over a field with $q$ elements. Let $\theta : SL(n, q) \to SL(n, q)$ be the automorphism of $SL(n, q)$ given by $\theta(A) = (A^t)^{-1}$, where $A^t$ denotes the transpose of the matrix $A \in SL(n, q)$. In this case one can define the split extension $SL(n, q).<\theta>$ that following the notations used in [6] is denoted by $K_3^2(q)$. Therefore we have $K_3^2(q) = \langle SL(n, q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} \rangle$, see [4].

Now let $G$ denote the group $SL(n, q)$ and let the split extension of $G$ by the cyclic group $<\theta>$ of order 2 be denoted by $G^+$. Since $[G^+ : G] = 2$, we have $G^+ = G \cup \theta G$, and elements of $G^+$ which lie in $G$ are called positive and those outside $G$ are called negative elements. A conjugacy class in $G^+$ is called positive if it lies in $G$ otherwise it is called negative. We may assume that using [7] one can obtain information about conjugacy classes and complex irreducible characters of $G$, therefore so far as conjugacy classes of $G^+$ are concerned one must pay attention to negative conjugacy classes of $G^+$.

One can show that there is a one-to-one correspondence between the set of negative conjugacy classes of $G^+$ and the set of equivalence classes of invertible matrices in $G$.

Now we begin with a summary of facts relevant to the irreducible complex characters of $K_3^2(q)$.

Complex irreducible characters of $G^+$ are divided into two kinds. The group $<\theta>$ acts on the set of complex irreducible characters of $G$ as follows. If $\chi \in Irr(G)$, then $\chi^\omega(A) := \chi(\theta^{-1}A\theta)$. If $\chi^\omega = \chi$, then we say that $\chi$ is invariant under $<\theta>$ and in this case $\chi$ forms an orbit of $G^+$ acting on $Irr(G)$. Now by standard results that can be found in [8] there exists an irreducible character $\varphi$ of $G^+$ such that $\varphi \downarrow_G = \chi$. Since $G^+/G \cong Z_2$ has two linear characters, therefore multiplication of $\varphi$ with the non-trivial character of $G^+/G$ gives another irreducible character $\varphi'$ of $G^+$ such that $\varphi \downarrow_G = \chi$. In this case we say that $\chi$ extends to $\varphi$ and $\varphi'$ and it is enough to calculate one of them on the negative conjugacy classes of $G^+$.

As we mentioned earlier we have $K_3^2(q) = SL(2, q).<\theta> \cong SL(2, q),\theta \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1}, \forall A \in SL(2, q)$. In the following Lemma we give the structure of $K_3^2(q)$.

Lemma 1: Let $G = K_3^2(q)$. If $q$ is even, then $K_3^2(q) \cong SL(2, q) \times <\theta>$ and if $q$ is odd, then $K_3^2(q) \cong SL(2, q) \circ 4$ a central product of $SL(2, q)$ with the cyclic group of order 4.

Proof. The automorphism $\theta : SL(2, q) \to SL(2, q)$ is
given by $\theta(A) = (A^{-1})^t$ for all $A \in SL(2, q)$. If we set $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then it is easy to verify that for $A \in SL(2, q)$ we have $J^{-1}AJ = (A^{-1})^t$ and therefore $\theta$ is equal to an inner automorphism of $SL(2, q)$. We have $K_2^2(q) \cong SL(2, q, \theta)$ and hence $K_2^2(q) \cong SL(2, q)$ and since $\theta J > 0$, then we find that $K_2^2(q) \cong SL(2, q)$. If the characteristic of $GF(q)$ is even, then we get $K_2^2(q) \cong SL(2, q) < \theta J > 0$, and if the characteristic of $GF(q)$ is odd we obtain $K_2^2(q) \cong SL(2, q) < \theta J >$ the central product of $SL(2, q)$ with a cyclic group of order 4.

By [5] we have two important lemmas as follows

**Lemma 2:** a) Let $V_i(i = 1, 2)$ be $KG$-modules. Then the tensor product $V_1 \otimes_K V_2$ over $K$ obviously becomes a $K[G_1 \times G_2]$ module by

$$(v_1 \otimes v_2)(g_1, g_2) = v_1(g_1) \otimes v_2(g_2)$$

For $v_i \in V_i, g_i \in G$ .

If $\chi_i$ is the character of $G_i$ on $V_i$, then the character $\tau$ of $G_1 \times G_2$ on $V_1 \otimes V_2$ is given by

$$\tau((g_1, g_2)) = \chi_1(g_1)\chi_2(g_2)$$

For $g_i \in G_i$ .

b) Let $\chi_1, ..., \chi_k$ be the irreducible characters of $G_i$ over $C$ and $\psi_1, ..., \psi_k$ the irreducible characters of $G_2$ over $C$. Then the $t_{ij}$ defined by $t_{ij}(g_1, g_2) = \chi_i(g_1)\psi_j(g_2)$ where $i = 1, ..., h$ and $j = 1, ..., k$ are all the irreducible characters of $G_1 \times G_2$ .

**Lemma 3:** Let $F$ be the finite field of $q = 2^n$ elements, and let $\nu$ be a generator of the cyclic group $F^* = F - 0$ . Denote

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

in $G = SL(2, F)$. $G$ contains an element b of order $q + 1$. For any $x \in G$, denote the conjugacy class of $G$ containing $x$. Then $G$ has exactly $q + 1$ conjugacy classes $(1), (c), (a), (a^2), ..., (a^{q-2/2}), (b), ..., (b^{q/2})$, where

![Table](image)

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>$c$</th>
<th>$a^l$</th>
<th>$b^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$([x])$</td>
<td>1</td>
<td>$q^t - 1$</td>
<td>$q(q + 1)$</td>
<td>$q(q - 1)$</td>
</tr>
</tbody>
</table>

for $1 \leq l \leq (q - 2)/2, 1 \leq m \leq q/2$. Let $\rho \in C$ be a primitive $(q - 1)$-th root of 1 , table of $G$ over $C$ is

**Table (2)**

Character Table of $SL(2, 2^n)$

<table>
<thead>
<tr>
<th>$1$</th>
<th>$c$</th>
<th>$a^l$</th>
<th>$b^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi$</td>
<td>$q$</td>
<td>$0$</td>
<td>$l$</td>
</tr>
<tr>
<td>$\chi_i$</td>
<td>$q + 1$</td>
<td>$\rho^m + \rho^{-n}$</td>
<td>0</td>
</tr>
<tr>
<td>$\theta_j$</td>
<td>$q - 1$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

for $1 \leq i \leq (q - 2)/2, 1 \leq j \leq q/2, 1 \leq I \leq (q - 2)/2, 1 \leq m \leq q/2$.

**Remark 1:** In the case of $q$ even $< \theta J >$ has order 2 and its irreducible characters are denoted by $\mu_0$ and $\mu_1$ where $\mu_0$ is the identity character. Regarding the structure of $K_2^2(q)$ and Lemmas 2, 3 the irreducible characters of $K_2^2(q)$ in the case of $q$ even are $\mu_k 1_G, \mu_k \psi_i, \mu_k \chi_i$ and $\mu_k \theta_j$ where $k = 0, 1$ and $1 \leq i \leq \frac{q - 2}{2}, 1 \leq j \leq \frac{q}{2}$.

**Lemma 4:** Let $G = SL(2, q)$, if $q$ is a power of 2 then the Schur index of any irreducible character of $G$ over the rational numbers $Q$ is 1.

**Proof.** See [9].

By [9] it is easy to see that:

**Lemma 5:** Let $G = H \times K$ and $\psi \in IRR(H)$ and $\theta \in IRR(K)$. Let $\chi = \psi \times \theta$ and let $F \subseteq C$.

a) $m_F(\chi)$ divides $m_F(\psi)m_F(\theta)$.

b) Equality occurs in (a) provided $(m_F(\psi), \psi(1)|F(\theta) : F|) = 1$ and $(m_F(\theta), \psi(1)|F(\psi) : F|) = 1$.

**Lemma 6:** Let $G$ be a finite group. If the Schur index of each non-principal irreducible character is equal to $m$, then $q(G) = m e(G)$.

**Proof.** See [1], Corollary 3.15.

We can see all the following statements in [1],[2].

**Definition 1:** Let $\chi$ be a complex character of $G$, such that $\ker \chi = 1$. Then define

1) $d(\chi) = |\Gamma(\chi)(x)(1)|$

2) $m(\chi) = \left\{ \begin{array}{ll} 0 & \chi = 1_G \\ |\min\{\sum_{\alpha \in I(\chi)} \chi^\alpha(g) : g \in G\}| & otherwise \end{array} \right.$

3) $c(\chi) = \sum_{\alpha \in I(\chi)} \chi^\alpha + m(\chi) 1_G$.

Now according to Corollary 3.11 of [1] and above statements the following lemma is useful for calculation of

$r(G), c(G)$ and $q(G)$.

**Lemma 7:** Let $G$ be a finite group with a unique minimal normal subgroup. Then

1) $r(G) = \min\{d(\chi) : \chi$ is a faithful irreducible complex character of $G$\}$

2) $c(G) = \min\{c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G$\}$
3) \( q(G) = \min \{ m_2(\chi) c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G \} \).

By [2] we have the following lemmas:

**Lemma 8**: Let \( \varepsilon \) be a primitive n-th root of unity in \( C \). Then \( \varepsilon + \varepsilon^{-1} \) is rational if and only if \( n = 1, 2, 3, 4, 6 \). The values which occur are as follows:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon^{1} )</td>
<td>2</td>
<td>-2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Lemma 9**: Let \( \varepsilon \) be a primitive n-th root of unity in \( C \) and \( m \in \mathbb{Z} \). If \( \varepsilon + \varepsilon^{-1} \) is rational, then \( \varepsilon^{m} + \varepsilon^{-m} \).

**Lemma 10**: Let \( \varepsilon \) be a primitive n-th root of unity. Then \( \varepsilon^{j} + \varepsilon^{-j}, 1 \leq j \leq n \) is rational if and only if \( n = 2j, 3j, 4j, 6j, \varepsilon^{j}, \varepsilon^{3j}, \frac{2j}{3} \).

**Lemma 11**: Let \( \varepsilon \in \text{ Irr}(G), \chi \neq 1_G \). Then \( c(\chi)(1) \geq d(\chi) + 1 \geq (\chi + 1) + 1 \).

**Proof**: From Definition 1 it follows that \( c(\chi)(1) \) is a non-negative real-valued character of \( G \) so by [1], Lemma 3.2, \( m(\chi) \geq 1 \). Now the result follows from Definition 1.

**Lemma 12**: Let \( \chi \in \text{ Irr}(G) \). Then
1. \( c(\chi)(1) \geq d(\chi) \geq (\chi) \);
2. \( c(\chi)(1) \leq 2d(\chi) \). Equality occurs if and only if \( \chi \text{ is of even order} \).

**Proof**: (1) follows from the definition of \( c(\chi)(1) \) and \( d(\chi) \).

**Lemma 13**: Let \( G = SL(2, q) \) where \( q = 2^n \) and \( n \geq 2 \).

Then for each \( j, 1 \leq j \leq q/2 \),
1. \( \theta_j \) is rational if and only if \( q \equiv -1 \mod 3 \) and \( j = \frac{q+1}{3} \);
2. \( d(\theta_j) \geq q - 1 \) and equality holds if \( \theta_j \) is rational.

**Proof**: As \( 1 \leq j \leq q/2 < \frac{q+1}{3} \) and as \( \sigma = \text{ a primitive } (q+1)-\text{root of unity} \), Lemmas 9 and 10 implies that \( \theta_j \) is rational if only if \( j = \frac{q+1}{4}, \frac{q+1}{4}, \frac{q+1}{8} \). Since \( q+1 \) is odd, \( \frac{q+1}{4} \) and \( \frac{q+1}{8} \) are not integers. Thus, \( \sigma^1 + \sigma^{-1} \in Q \) if and only if \( 3|q+1 \) and \( j = \frac{q+1}{3} \). This proves (1). If \( \theta_j \) is not rational, then \( |\Gamma| \geq 2 \) where \( \Gamma = \Gamma(Q(\theta_j); Q) \) so that \( c(\theta_j)(1) \geq d(\theta_j) \geq (q-1) + 1 \) by Lemma 12. On the other hand if \( 3|q+1 \), then \( 8 \leq q \), so that \( 3 < \frac{q}{2} \); but \( \theta_j (b_3) = -2 < \theta_j (g) \) for all \( g \in G \) so that \( m(\theta_j) = 2 \). Thus \( d(\theta_j) = q - 1 \) and \( c(\theta_j)(1) = q + 1 \). This completes the proofs of (2) and (3).

**Theorem 2**: Let \( G = K_2(2) \) then \( r(G) = 2, c(G) = q(G) = 4 \).

**Proof**: By Lemmas 4, 5 Schur index of each irreducible characters is 1 and so by Lemma 6 we have \( c(G) = q(G) \).

Since the only faithful irreducible character of \( G \) is \( \mu_1 = \psi \) by the character table of the group \( K_2(2) \) the result follows.

<table>
<thead>
<tr>
<th>( \chi ) (faithful)</th>
<th>( d(\chi) )</th>
<th>( c(\chi)(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_1\psi )</td>
<td>( q )</td>
<td>( 2q )</td>
</tr>
<tr>
<td>( \mu_1x_i )</td>
<td>( q+1 )</td>
<td>( 2(q+1) )</td>
</tr>
<tr>
<td>( \mu_1\theta_j )</td>
<td>( q-1 )</td>
<td>( 2(q-1) )</td>
</tr>
</tbody>
</table>

Now by Lemma 7, Lemma 13 and Table (4) when \( q \equiv -1 \mod 3 \) we have

\[ \min \{ d(\chi) : \chi \text{ is a faithful irreducible complex character of } G \} = q - 1 \] and

\[ \min \{ c(\chi)(1) : \chi \text{ is a faithful irreducible complex character of } G \} = 2(q-1) \] Otherwise \( d(\mu_1\theta_j) > q - 1 \) and so in this case \( d(\chi) = q \) and \( c(\chi)(1) = 2q \).