Intuitionistic T-S Fuzzy Subalgebras and Ideals in BCI-algebras

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Abstract—The aim of this paper is to introduce the notions of intuitionistic T-S fuzzy subalgebras and intuitionistic T-S fuzzy ideals in BCI-algebras, and then to investigate their basic properties.

Keywords—BCI-algebra, intuitionistic T-S fuzzy subalgebra, intuitionistic T-S fuzzy ideal, norm intersection, direct product, epimorphism, isomorphism.

I. INTRODUCTION

A BCI-algebra is an important class of logical algebras, the theory of BCI-algebras introduced by Iséki [1] has been studied deep by several researchers so far. Xi [2] introduced the concepts of fuzzy subalgebras and ideas in BCI-algebras and discussed some properties of them. On the other hand, triangular norm is a powerful tool in the theory research and application development of fuzzy sets. On the basis of the definition of the intuitionistic fuzzy groups, Li [4], [5] generalized the operators “∧” and “∨” to T-norm and S-norm and defined the intuitionistic fuzzy groups of T-norms. In this paper, the concepts of intuitionistic T-S fuzzy subalgebras and intuitionistic T-S fuzzy ideas are introduced in BCI-algebras. Some properties are discussed. We prove that the norm intersection and direct product of two intuitionistic T-S fuzzy subalgebras (ideals) are also intuitionistic T-S fuzzy subalgebras (ideals) in BCI-algebras.

II. PRELIMINARIES

An algebra \((X;\star,0)\) of type \((2,0)\) is called a BCI-algebra if it satisfies the following axioms:

- **(BCI-1)** \(x\star(x\star y)\star(x\star z)\star(x\star y) = 0\),
- **(BCI-2)** \((x\star(x\star y))\star y = 0\),
- **(BCI-3)** \(x\star x = 0\),
- **(BCI-4)** \(x\star y = 0\) and \(y\star x = 0\) imply \(x = y\),

for all \(x, y, z \in X\). In a BCI-algebra \(X\), we can define a partial ordering \(\leq\) by putting \(x \leq y\) if and only if \(x\star y = 0\).

In this paper, \(X\) always means a BCI-algebra unless otherwise specified.

**Definition 1** [3] Let \(S\) be any set. An intuitionistic fuzzy subset \(A\) of \(S\) is an object of the following form

\[
A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in S \} \text{, where } \mu_A : S \rightarrow [0,1] \text{ and } \nu_A : S \rightarrow [0,1]
\]

and define the degree of membership and the degree of non-membership of the element \(x \in S\) respectively and for every \(x \in S\), \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\).

The all intuitionistic fuzzy subsets of \(S\) be denote \(IFS[S]\).

**Definition 2** [4] Mapping \(T : [0,1] \times [0,1] \rightarrow [0,1]\) is called as triangular norm, if \(T\) satisfies:

\[ T(0,0) = 0, T(1,1) = 1, \]

\[ T(a, b) \geq T(c, d) \text{ if } a \geq c, b \geq d; \]

\[ T(a, b) = T(b, a); \]

\[ T(a, T(b, c)) = T(T(a, b), c). \]

When the triangular \(T\) satisfies \(T(a,1) = a\), it is called as T-norm; When \(T\) satisfies \(T(a,0) = a\), it is called as S-norm.

Specifically, let \(T_0 = x \land y\) and \(S_0 = x \lor y\) for all \(x, y \in [0,1]\), we have \(T \leq T_0 \leq S_0 \leq S\).

**Definition 3** [4] Let \(x' = 1-x\) for all \(x \in [0,1]\), we say that \(x'\) is a complement of \(x\). Given T-norm \(T(x,y)\), let \(S(x,y)=1-T(1-x,1-y)=T(x',y')\), obviously, \(S(x,y)\) is an S-norm. We say that \(S\) and \(T\) are dual.

**Definition 4** [7] Let \(X, Y\) be two nonempty classical sets and \(A \in IFS[X], B \in IFS[Y]\). Let the mapping \(f : X \rightarrow Y\), then the mapping \(f\) can induce a mapping \(F_f\) from \(IFS[X]\) to \(IFS[Y]\) and \(F_f^{-1}\) from \(IFS[Y]\) to \(IFS[X]\),

\[
F_f(A) = \{ \langle x, \mu_{F_f(a)}(x), \nu_{F_f(a)}(y) \rangle : y \in Y \},
\]

\[
F_f^{-1}(B) = \{ \langle x, \mu_{F_f^{-1}(a)}(x), \nu_{F_f^{-1}(a)}(y) \rangle : x \in X \},
\]

where

\[
\mu_{F_f(a)}(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu_A(x), & f^{-1}(y) \neq \emptyset \\
0, & f^{-1}(y) = \emptyset 
\end{cases},
\]

\[
\nu_{F_f(a)}(y) = \begin{cases} 
\inf_{x \in f^{-1}(y)} \nu_A(x), & f^{-1}(y) \neq \emptyset \\
1, & f^{-1}(y) = \emptyset 
\end{cases},
\]

\[
\mu_{F_f^{-1}(a)}(x) = \mu_A(f(x)), \nu_{F_f^{-1}(a)}(x) = \nu_A(f(x)).
\]

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Proposition 1 [4] Let $T$ be a T-norm and $S$ be an S-norm, for all $T$ and $S$, if $a,b,c,d \in [0,1]$, then
\[ T(T(a,b),T(c,d)) = T(T(a,c),T(b,d)) \]
\[ = T(T(a,d),T(b,c)); \]
(2) $S(S(a,b),S(c,d)) = S(S(a,c),S(b,d))$.

III. INTUITIONISTIC T-S FUZZY SUBALGEBRAS IN BCI-ALGEBRAS

Definition 5 Let $T$ be a T-norm, $S$ be an S-norm and $T$ and $S$ be dual norm. An intuitionistic fuzzy set $A$ in $X$ is called an intuitionistic T-S fuzzy subalgebra of $X$ if the following are satisfied:
\[ \mu_x (x^y) \geq T(\mu_x(x),\mu_y(y)), \]
\[ \nu_x (x^y) \leq S(\nu_x(x),\nu_y(y)), \]
for all $x, y \in X$.

Proposition 2 Let $T$ be a T-norm, $S$ be an S-norm and $T$ and $S$ be dual norm. Let $A = \{ (x,\mu_x(x),\nu_x(x)) : x \in X \}$ be an intuitionistic T-S fuzzy subalgebra of $X$, then
\[ \square A = \{ (x,\mu_x(x),1-\mu_x(x)) : x \in X \}, \]
\[ \triangle A = \{ (x,0,\nu_x(x)) : x \in X \} \]
are both intuitionistic T-S fuzzy subalgebras of $X$.

Proof. Denote $\omega_x(x) = 1-\mu_x(x)$, for all $x, y \in X$, we have
\[ \omega_x(x^y) = 1-\mu_x(x) \]
\[ \leq 1-T(\mu_x(x),\mu_y(y)) \]
\[ = T(T(\mu_x(x),\mu_y(y)) \]
\[ = S(S(\mu_x(x),\mu_y(y)) \]
\[ = S(1-\mu_x(x),1-\mu_y(y)) \]
\[ = S(\omega_x(x),\omega_y(y)). \]
Thus
\[ \square A = \{ (x,\mu_x(x),1-\mu_x(x)) : x \in X \} \]
is an intuitionistic T-S fuzzy subalgebra of $X$.

Denote $\gamma_x(x) = 1-\nu_x(x)$, for all $x, y \in X$, we have
\[ \gamma_x(x^y) = 1-\nu_x(x) \]
\[ \geq 1-S(\nu_x(x),\nu_y(y)) \]
\[ = S(S(\nu_x(x),\nu_y(y)) \]
\[ = T(T(\nu_x(x),\nu_y(y)) \]
\[ = T(1-\nu_x(x),1-\nu_y(y)) \]
\[ = T(\gamma_x(x),\gamma_y(y)). \]
Thus
\[ \triangle A = \{ (x,1-\nu_x(x),\nu_x(x)) : x \in X \} \]
is also an intuitionistic T-S fuzzy subalgebra of $X$.

Definition 6 [8] Let $X$ be any set, $A \in IFS[X]$, for all $\lambda \in [0,1]$, $\lambda A = \{ (x,\lambda \mu_x(x),\lambda \nu_x(x)) : x \in X \}$, where
\[ \lambda \mu_x(x) = \left\{ \begin{array}{ll}
\mu_x(x), & \lambda \geq \mu_x(x) \\
\lambda, & \lambda < \mu_x(x)
\end{array} \right\}
\]
$\lambda \nu_x(x) = \left\{ \begin{array}{ll}
\nu_x(x), & \lambda \geq \nu_x(x) \\
\lambda, & \lambda < \nu_x(x)
\end{array} \right\}$.
$\lambda A$ is called as cutproduct of $\lambda$ and $A$.

Proposition 3 Let $A$ be an intuitionistic T-S fuzzy subalgebra of $X$, then for any $\lambda \in [0,1]$, $\lambda A$ is also an intuitionistic T-S fuzzy subalgebra of $X$.

Proof. It is clear that $\lambda A \in IFS[X]$. In the following we need to verify that $\lambda A$ satisfies the conditions of Definition 5.

If $\lambda \geq \mu_x(x^y)$, then
\[ \lambda \mu_x(x^y) = \mu_x(x^y) \]
\[ \geq T(\mu_x(x),\mu_y(y)) \]
\[ \geq T(\lambda \mu_x(x),\lambda \mu_y(y)). \]
If $\lambda < \mu_x(x^y)$, then
\[ \lambda \mu_x(x^y) = \lambda = T(\lambda,1) \geq T(\lambda \mu_x(x),\lambda \mu_y(y)) \];

hence we can obtain $\lambda \mu_x(x^y) \geq T(\lambda \mu_x(x),\lambda \mu_y(y))$ for any $\lambda \in [0,1]$.

If $\lambda \geq \nu_x(x^y)$ and $\lambda \geq \max \{ \nu_x(x),\nu_y(y) \}$, then
\[ \lambda \nu_x(x^y) = \nu_x(x^y) \]
\[ \leq S(\nu_x(x),\nu_y(y)) \]
\[ = S(\lambda \nu_x(x),\lambda \nu_y(y)). \]
If $\lambda \geq \nu_x(x^y)$ and $\lambda \leq \max \{ \nu_x(x),\nu_y(y) \}$, then
\[ S(\lambda \nu_x(x),\lambda \nu_y(y)) \geq S(\lambda,0) = S(0,\lambda) = \lambda, \]
\[ \lambda \nu_x(x^y) = \nu_x(x^y) \leq \lambda \leq S(\lambda \nu_x(x),\lambda \nu_y(y)); \]

If $\lambda < \nu_x(x^y)$ and $\lambda \geq \max \{ \nu_x(x),\nu_y(y) \}$, then
\[ \lambda \nu_x(x^y) = \lambda \]
\[ < \nu_x(x^y) \]
\[ \leq S(\nu_x(x),\nu_y(y)) \]
\[ = S(\lambda \nu_x(x),\lambda \nu_y(y)); \]
If $\lambda < \nu_x(x^y)$ and $\lambda \leq \max \{ \nu_x(x),\nu_y(y) \}$, then
\[ S(\lambda \nu_x(x),\lambda \nu_y(y)) \geq S(\lambda,0) = S(0,\lambda) = \lambda, \]
\[ \lambda \nu_x(x^y) = \lambda \leq S(\lambda \nu_x(x),\lambda \nu_y(y)); \]
Summarizing the above arguments, \(\lambda A\) is an intuitionistic T-S fuzzy subalgebra of \(X\).

**Definition 7** [6] If \(A = \{(x, \mu_A(x), \nu_A(x)) : x \in S\}\) and \(B = \{(x, \mu_B(x), \nu_B(x)) : x \in S\}\) be any two intuitionistic fuzzy subsets of a set \(S\), then
\[
A \cap B = \{(x, \mu_{A \cap B}(x), \nu_{A \cap B}(x)) : x \in S\}
\]
where
\[
\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x),
\nu_{A \cap B}(x) = \mu_A(x) \vee \mu_B(x),
\]

**Proposition 4** Let \(A \) and \(B\) be two intuitionistic T-S fuzzy subalgebras of \(X\), then \(A \cap B\) is also an intuitionistic T-S fuzzy subalgebra of \(X\).

**Proof.** For all \(x, y \in X\). By Definition 5, Definition 7 and Proposition 1, we have
\[
\mu_{A \times B}(x \times y) = T(\mu_A(x), \mu_B(y)),
\nu_{A \times B}(x \times y) = S(\nu_A(x), \nu_B(y))
\]

Hence \(A \times B\) is an intuitionistic T-S fuzzy subalgebra of \(X\).

**Definition 8** [6] Let \(A \) and \(B\) be two intuitionistic fuzzy sets of a set \(X\).

The Cartesian product of \(A\) and \(B\) is defined by
\[
A \times B = \{(x, \mu_A(x), \nu_A(x)) : x \in X \times X\}
\]
where
\[
\mu_{A \times B}(x) = T(\mu_A(x), \mu_B(x)),
\nu_{A \times B}(x) = S(\nu_A(x), \nu_B(x))
\]

**Proposition 5** Let \(A\) and \(B\) be two intuitionistic T-S fuzzy subalgebras of \(X\), then \(A \times B\) is also an intuitionistic T-S fuzzy subalgebra of \(X \times X\).

**Proof.** For all \(x = (x_1, x_2), y = (y_1, y_2) \in X \times X\), by Definition 5, Definition 8 and Proposition 1, we get
\[
\mu_{A \times B}(x \times y) = \mu_{A \times B}(x_1 \times x_2, y_1 \times y_2)
\]
\[
\nu_{A \times B}(x \times y) = \nu_{A \times B}(x_1 \times x_2, y_1 \times y_2)
\]

IV. INTUITIONISTIC T-S FUZZY IDEALS IN BCI-ALGEBRAS

**Definition 9** Let \(T\) be a T-norm, \(S\) be an S-norm and \(T\) and \(S\) be dual norm. An intuitionistic fuzzy set \(A\) in \(X\) is called an intuitionistic T-S fuzzy ideal of \(X\) if the following are satisfied:
\[
(F_1) \mu_A(0) \geq \mu_A(x),
(F_2) \mu_A(x) \geq T(\mu_A(x), \mu_A(y)),
(F_3) \nu_A(0) \leq \nu_A(x),
(F_4) \nu_A(x) \leq S(\nu_A(x), \nu_A(y)),
\]
for all \(x, y \in X\).

**Proposition 6** Let \(A\) be an intuitionistic T-S fuzzy ideal of \(X\) and \(\mu_A(0) = 1\), \(\nu_A(0) = 0\). If \(x \leq y\) holds in \(X\), then
\[
\mu_A(x) \geq \mu_A(y), \quad \nu_A(x) \leq \nu_A(y).
\]

**Proof.** For all \(x, y \in X\), if \(x \leq y\), then \(x \times y = 0\), so by Definition 8,
\[
T(\mu_A(x), \mu_A(y)) = \mu_A(x),
S(\nu_A(x), \nu_A(y)) = \nu_A(x).
\]

**Proposition 7** Let \(A\) be an intuitionistic T-S fuzzy ideal of \(X\), \(\mu_A(0) = 1\) and \(\nu_A(0) = 0\). If the inequality \(x \times y \leq z\) holds in \(X\), then
\[
\mu_A(x) \geq T(\mu_A(x), \mu_A(y)), \quad \nu_A(x) \leq S(\nu_A(y), \nu_A(z)).
\]

**Proof.** For all \(x, y, z \in X\), if \(x \times y \leq z\), then
\[
\mu_A(x) \geq T(\mu_A(x), \mu_A(y)), \quad \nu_A(x) \geq S(\nu_A(y), \nu_A(z)).
\]

**Proposition 8** Let \(A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}\) be an intuitionistic T-S fuzzy ideal of \(X\), then
\[
\bigwedge A = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in X\}.
\]
\( \emptyset A = \{ (x, 1 - \nu_y(x), \nu_y(x)) : x \in X \} \)

are both intuitionistic T-S fuzzy ideals of \( X \).

**Proof.** Denote \( \omega_x(x) = 1 - \mu_x(x) \), for all \( x, y \in X \), we have
\[
\omega_x(0) = 1 - \mu_x(0) \\
\leq 1 - \mu_x(x) \\
= \omega_x(x), \\
\omega_x(x) = 1 - \mu_x(x) \\
\leq 1 - T(\mu_x(x \ast y), \mu_y(y)) \\
= T(\mu_x(x \ast y), \mu_y(y)) \\
= S(\mu_x(x \ast y), \mu_y(y)) \\
= S(1 - \mu_x(x \ast y), 1 - \mu_y(y)) \\
= S(\omega_x(x \ast y), \omega_y(y)).
\]

Thus
\[
\emptyset A = \{ (x, \mu_x(x), 1 - \mu_x(x) : x \in X \} 
\]

is an intuitionistic T-S fuzzy ideal of \( X \).

**Proposition 9** Let \( A \) and \( B \) be two intuitionistic T-S fuzzy ideals of \( X \), then \( A \cap B \) is also an intuitionistic T-S fuzzy ideal of \( X \).

**Proof.** For all \( x \in X \). By Definition 7, Definition 9 and Proposition 1, we have
\[
\mu_{\ast \cap} (0) = T(\mu_x(0), \mu_y(0)) \\
\geq T(\mu_x(x), \mu_y(y)) \\
= \mu_{\ast \cap}(x), \\
\nu_{\ast \cap} (0) = S(\nu_x(0), \nu_y(0)) \\
\leq S(\nu_x(x), \nu_y(y)) \\
= \nu_{\ast \cap}(x).
\]

Thus
\[
\emptyset A = \{ (x, 1 - \nu_y(x), \nu_y(x)) : x \in X \}
\]

is also an intuitionistic T-S fuzzy ideal of \( X \).

**Proposition 10** Let \( A \) and \( B \) be two intuitionistic T-S fuzzy ideals of \( X \), then \( A \ast B \) is also an intuitionistic T-S fuzzy ideal of \( X \times X \).

**Proof.** For all \( x = (x_1, x_2), y = (y_1, y_2) \in X \times X \), by Definition 8, Definition 9 and Proposition 1, we get
\[
\mu_{\ast \ast} (0) = T(\mu_x(0), \mu_y(0)) \\
\geq T(\mu_x(x_1), \mu_y(y_1)) \\
= \mu_{\ast \ast}(x), \\
\nu_{\ast \ast} (0) = S(\nu_x(0), \nu_y(0)) \\
\leq S(\nu_x(x_1), \nu_y(y_1)) \\
= \nu_{\ast \ast}(x), \\
\mu_{\ast \ast}(x) = T(\mu_x(x_1), \mu_y(y_1)) \\
\geq T(\mu_x(x_1 \ast y_1), \mu_y(y_1)) \\
= T(\mu_x(x_1 \ast y_1), \mu_y(y_1)) \\
= S(\nu_x(x_1 \ast y_1), \nu_y(y_1)) \\
= S(\nu_x(x_1 \ast y_1), \nu_y(y_1)) \\
= S(\nu_{\ast \ast}(x_1 \ast y_1), \nu_{\ast \ast}(y_1)).
\]

Hence \( A \ast B \) is also an intuitionistic T-S fuzzy ideal of \( X \times X \).

**Proposition 11** If \( f : X \rightarrow X' \) is an epimorphism from \( X \) to \( X' \) and \( B \) is an intuitionistic T-S fuzzy ideal of \( X' \), then \( F^{-1}_f(B) \) is an intuitionistic T-S fuzzy ideal of \( X \).

**Proof.** Let \( B = \{ (x', \mu_y(x'), \nu_y(x')) : x' \in X' \} \) be an intuitionistic T-S fuzzy ideal of \( X' \), then
\[
F^{-1}_f(B) = \{ (x, \mu_y^{-1}(x), \nu_y^{-1}(x)) : x \in X \}.
\]
Since $f : X \to X'$ is an epimorphism from $X$ to $X'$, therefore $F_f(0) = 0'$. Let $x', y' \in X'$, then exist $x, y \in X$, such that $x' = F_f(x), y' = F_f(y)$. Let $F_f^{-1}(x')*F_f^{-1}(y') = \{x*y : x \in F_f^{-1}(x'), y \in F_f^{-1}(y')\}$, then $F_f^{-1}(x')*F_f^{-1}(y') \subseteq F_f^{-1}(x'*y')$.

we get $F_f(x*y) = x'*y'$.

$$\mu_{F_f^{-1}(0)}(0) = \mu_b(F_f(0))$$

= $\mu_b(0')$ 

$\geq \mu_b(x')$ 

= $\mu_{F_f^{-1}(0)}(x)$, 

$$v_{F_f^{-1}(0)}(0) = v_b(F_f(0))$$

= $v_b(0')$ 

$\leq v_b(x')$ 

= $v_{F_f^{-1}(0)}(x)$, 

$$\mu_{F_f^{-1}(x)}(x) = \mu_b(F_f(x))$$

= $\mu_b(x')$ 

$\geq T(\mu_b(x'*y'), \mu_b(y'))$ 

= $T(\mu_{F_f^{-1}(x)}(x'*y'), \mu_{F_f^{-1}(y)}(y'))$. 

$$v_{F_f^{-1}(x)}(x) = v_b(F_f(x))$$

= $v_b(x')$ 

$\leq S(v_b(x'*y'), v_b(y'))$ 

= $S(v_{F_f^{-1}(x)}(x'*y'), v_{F_f^{-1}(y)}(y'))$. 

Hence $F_f^{-1}(B)$ is an intuitionistic T-S fuzzy ideal of $X$.

**Proposition 12** If $f : X \to Y$ is an isomorphism mapping from $X$ to $Y$ and $A, B$ are two intuitionistic T-S fuzzy ideals of $X$ and $Y$ respectively then

1. $F_f^{-1}(F_f(A)) = A$,
2. $F_f(F_f^{-1}(B)) = B$.

**Proof.** (1) Since $f$ is an isomorphism mapping from $X$ to $Y$, for all $x \in X$, let $y = f(x)$, then $f^{-1}(y) = \{x\}$ and

$$\mu_{F_f^{-1}(f^{-1}(y))}(x) = \mu_{F_f^{-1}(f^{-1}(y))}(y)$$

= $\mu_A(x) = \mu_A(x)$.

By the similar proof ways, we can have $v_{F_f^{-1}(f^{-1}(y))}(x) = v_A(x)$, for every $x \in X$.

This implies that $F_f^{-1}(F_f(A)) = A$.

(2) It is clear that $F_f(F_f^{-1}(B))$ is an intuitionistic T-S fuzzy ideal of $Y$, for all $y \in Y$, we have

$$\mu_{F_f(F_f^{-1}(B))}(y) = \sup_{x \in f^{-1}(y)} \mu_{F_f^{-1}(B)}(x) = \mu_b(f(x)) = \mu_b(y)$$. By the similar proof ways, we can have $v_{F_f(F_f^{-1}(B))}(y) = v_A(y)$, for every $x \in X$.

This implies that $F_f(F_f^{-1}(B)) = B$.

**REFERENCES**


