Intuitionistic T-S Fuzzy Subalgebras and Ideals in BCI-algebras

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Abstract—The aim of this paper is to introduce the notions of intuitionistic T-S fuzzy subalgebras and intuitionistic T-S fuzzy ideals in BCI-algebras, and then to investigate their basic properties.

Keywords—BCI-algebra, intuitionistic T-S fuzzy subalgebra, intuitionistic T-S fuzzy ideal, norm intersection, direct product, epimorphism, isomorphism.

I. INTRODUCTION

A BCI-algebra is an important class of logical algebras, the theory of BCI-algebras introduced by Iséki [1] has been studied deep by several researchers so far. Xi [2] introduced the concepts of intuitionistic fuzzy subalgebras and ideals in BCI-algebras and discussed some properties of them. On the other hand, triangular norm is a powerful tool in the theory research and application development of fuzzy sets. On the basis of the definition of the intuitionistic fuzzy groups, Li [4], [5] generalized the operators \( \wedge \) and \( \vee \) to T-norm and S-norm and defined the intuitionistic fuzzy groups of T-norms. In this paper, the concepts of intuitionistic T-S fuzzy subalgebras and intuitionistic T-S fuzzy ideals are introduced in BCI-algebras. Some properties are discussed. We prove that the norm intersection and direct product of two intuitionistic T-S fuzzy subalgebras (ideals) are also intuitionistic T-S fuzzy subalgebras (ideals) in BCI-algebras.

II. PRELIMINARIES

An algebra \((X;\ast,0)\) of type \((2,0)\) is called a BCI-algebra if it satisfies the following axioms:

- \((BCI-1)\) \((x\ast y)\ast(x\ast z)=(x\ast y)\ast z\)
- \((BCI-2)\) \((x\ast(x\ast y))\ast y=0\),
- \((BCI-3)\) \(x\ast x=0\),
- \((BCI-4)\) \(x\ast y=0\) and \(y\ast x=0\) imply \(x=y\),

for all \(x, y, z \in X\). In a BCI-algebra \(X\), we can define a partial ordering \(\leq\) by putting \(x \leq y\) if and only if \(x\ast y=0\).

In this paper, \(X\) always means a BCI-algebra unless otherwise specified.

Definition 1 [3] Let \(S\) be any set. An intuitionistic fuzzy subset \(A\) of \(S\) is an object of the following form

\[ A = \{ (x, \mu_A(x), \nu_A(x)) : x \in S \} \]

where \(\mu_A : S \rightarrow [0,1]\) and \(\nu_A : S \rightarrow [0,1]\) define the degree of membership and the degree of non-membership of the element \(x \in S\) respectively and for every \(x \in S\), \(0 \leq \mu_A(x) + \nu_A(x) \leq 1\).

The all intuitionistic fuzzy subsets of \(S\) be denote \(IFS[S]\).

Definition 2 [4] Mapping \(T : [0,1]\times[0,1] \rightarrow [0,1]\) is called as triangular norm, if \(T\) satisfies:

- \((T-1)\) \(T(0,0)=0\), \(T(1,1)=1\);
- \((T-2)\) \(T(a,b) \geq T(c,d)\) if \(a \geq c, b \geq d\);
- \((T-3)\) \(T(a,b)=T(b,a)\);
- \((T-4)\) \(T(a,T(b,c))=T(T(a,b),c)\).

When the triangular \(T\) satisfies \(T(a,1)=a\), it is called as T-norm; When \(T\) satisfies \(T(a,0)=a\), it is called as S-norm.

Specifically, let \(T_0=x\wedge y\) and \(S_0=x\vee y\) for all \(x, y \in [0,1]\), we have \(T_0 \leq T \leq S_0 \leq S\).

Definition 3 [4] Let \(x'=1-x\) for all \(x \in [0,1]\), we say that \(x'\) is a complement of \(x\). Given T-norm \(T(x,y)\), let \(S(x,y)=1-T(1-x,1-y)=T(x',y')\)

obviously, \(S(x,y)\) is an S-norm. We say that \(S\) and \(T\) are dual.

Definition 4 [7] Let \(X, Y\) be two nonempty classical sets and \(A \in IFS[X], B \in IFS[Y]\). Let the mapping \(f : X \rightarrow Y\), then the mapping \(f\) can induce a mapping \(F_f\) from \(IFS[X]\) to \(IFS[Y]\) and \(F_f^{-1}\) from \(IFS[Y]\) to \(IFS[X]\).

\[ F_f(A) = \{ (y, \mu_{F_f^{-1}}(y)) : y \in Y \} \]
\[ F_f^{-1}(B) = \{ (x, \mu_{F_f^{-1}}(x), \nu_{F_f^{-1}}(x)) : x \in X \} \]

where

\[ \mu_{F_f^{-1}}(y) = \sup \{ \inf_{x \in f^{-1}(y)} \mu_A(x) : x \in X \} \]
\[ \nu_{F_f^{-1}}(y) = 1 - \sup \{ \inf_{x \in f^{-1}(y)} \nu_A(x) : x \in X \} \]

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Proposition 1 [4] Let \( T \) be a T-norm and \( S \) be an S-norm, for all \( T \) and \( S \), if \( a, b, c, d \in [0,1] \), then
\[
(1) \ T (T(a,b),T(c,d)) = T(T(a,c),T(b,d)) = T(T(a,d),T(b,c));
\]
\[
(2) \ S(S(a,b),S(c,d)) = S(S(a,c),S(b,d)) = S(S(a,d),S(b,c)).
\]

III. INTUITIONISTIC T-S FUZZY SUBALGEBRAS IN BCI-ALGEBRAS

Definition 5 Let \( T \) be a T-norm, \( S \) be an S-norm and \( T \) and \( S \) be dual norm. An intuitionistic fuzzy set \( A \) in \( X \) is called an intuitionistic T-S fuzzy subalgebra of \( X \) if the following are satisfied:

\[
\mu_a(x,y) \geq T(\mu_a(x),\mu_a(y)), \quad \nu_a(x,y) \leq S(\nu_a(x),\nu_a(y)),
\]

for all \( x, y \in X \).

Proposition 2 Let \( T \) be a T-norm, \( S \) be an S-norm and \( T \) and \( S \) be dual norm. Let \( A = \{ (x, \mu_a(x), \nu_a(x)) : x \in X \} \) be an intuitionistic T-S fuzzy subalgebra of \( X \), then

\[
\square A = \{ (x, \mu_a(x), 1 - \mu_a(x)) : x \in X \},
\]

\[
\square A = \{ (x, 1 - \nu_a(x), \nu_a(x)) : x \in X \}
\]

are both intuitionistic T-S fuzzy subalgebras of \( X \).

Proof. Denote \( \omega_a(x) = 1 - \mu_a(x) \), for all \( x, y \in X \), we have

\[
\omega_a(x,y) = 1 - \mu_a(x,y)
\]

\[
\leq 1 - T(\mu_a(x), \mu_a(y)) = T(\mu_a(x), \mu_a(y))^{-1}
\]

\[
= S(1 - \mu_a(x), 1 - \mu_a(y)) = S(\omega_a(x), \omega_a(y)).
\]

Thus

\[
\square A = \{ (x, \mu_a(x), 1 - \mu_a(x)) : x \in X \} \text{ is an intuitionistic T-S fuzzy subalgebra of } X.
\]

Denote \( \gamma_a(x) = 1 - \nu_a(x) \), for all \( x, y \in X \), we have

\[
\gamma_a(x,y) = 1 - \nu_a(x,y)
\]

\[
\geq 1 - S(\nu_a(x), \nu_a(y)) = S(\nu_a(x), \nu_a(y))^{-1}
\]

\[
= T(\nu_a(x), \nu_a(y))^{-1} = T(1 - \nu_a(x), 1 - \nu_a(y))^{-1}.
\]

Thus

\[
\square A = \{ (x, \nu_a(x), 1 - \nu_a(x)) : x \in X \} \text{ is also an intuitionistic T-S fuzzy subalgebra of } X.
\]

Definition 6 [8] Let \( X \) be any set, \( A \in IFS[X] \), for all \( \lambda \in [0,1] \), \( \lambda A = \{ (x, \lambda \mu_a(x), \lambda \nu_a(x)) : x \in X \} \),

where

\[
\lambda \mu_a(x) = \mu_a(x) \lambda \geq \mu_a(x), \quad \lambda \nu_a(x) = \nu_a(x) \lambda \leq \nu_a(x).
\]

\( \lambda A \) is called as cut product of \( \lambda \) and \( A \).

Proposition 3 Let \( A \) be an intuitionistic T-S fuzzy subalgebra of \( X \), then for any \( \lambda \in [0,1] \), \( \lambda A \) is also an intuitionistic T-S fuzzy subalgebra of \( X \).

Proof. It is clear that \( \lambda A \in IFS[X] \). In the following we need to verify that \( \lambda A \) satisfies the conditions of Definition 5.

If \( \lambda \geq \mu_a(x,y) \), then

\[
\lambda \mu_a(x,y) = \mu_a(x,y) \geq \mu_a(x,y)
\]

\[
\lambda \nu_a(x,y) = \nu_a(x,y) \leq \nu_a(x,y).
\]

Thus

\[
\lambda A = \{ (x, \lambda \mu_a(x), \lambda \nu_a(x)) : x \in X \} \text{ is an intuitionistic T-S fuzzy subalgebra of } X.
\]

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Summarizing the above arguments, $\lambda A$ is an intuitionistic T-S fuzzy subalgebra of $X$.

**Definition 7** [6] If $A = \{ (x, \mu_A(x), \nu_A(x)) : x \in S \}$ and $B = \{ (x, \mu_B(x), \nu_B(x)) : x \in S \}$ be any two intuitionistic fuzzy subsets of a set $S$, then

$$A \cap B = \{ (x, \mu_{A \cap B}(x), \nu_{A \cap B}(x)) : x \in S \}$$

$$= \{ (x, T(\mu_A(x), \mu_B(x))S(\nu_A(x), \nu_B(x)) : x \in S \}.$$

**Proposition 4** Let $A$ and $B$ be two intuitionistic T-S fuzzy subalgebras of $X$, then $A \cap B$ is also an intuitionistic T-S fuzzy subalgebra of $X$.

**Proof.** For all $x, y \in X$. By Definition 5, Definition 7 and Proposition 1, we have

$$\mu_{A \cap B}(x \ast y) = T(\mu_A(x \ast y), \mu_B(x \ast y))$$

$$\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y)))$$

$$= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y)))$$

$$= T(\mu_{A \cap B}(x), \mu_{A \cap B}(y)),$$

$$\nu_{A \cap B}(x \ast y) = S(\nu_A(x \ast y), \nu_B(x \ast y))$$

$$\leq S(S(\nu_A(x), \nu_A(y)), S(\nu_B(x), \nu_B(y)))$$

$$= S(S(\nu_A(x), \nu_B(x)), S(\nu_A(y), \nu_B(y)))$$

$$= S(\nu_{A \cap B}(x), \nu_{A \cap B}(y)).$$

Hence $A \cap B$ is an intuitionistic T-S fuzzy subalgebra of $X$.

**Definition 8** [6] Let $A$ and $B$ be two intuitionistic fuzzy sets of a set $X$.

The Cartesian product of $A$ and $B$ is defined by

$$A \times B = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \times X \},$$

where

$$\mu_{A \times B}(x \ast y) = T(\mu_A(x), \mu_B(y)),$$

$$\nu_{A \times B}(x \ast y) = S(\nu_A(x), \nu_B(y)).$$

**Proposition 5** Let $A$ and $B$ be two intuitionistic T-S fuzzy subalgebras of $X$, then $A \times B$ is also an intuitionistic T-S fuzzy subalgebra of $X \times X$.

**Proof.** For all $x = (x_1, x_2), y = (y_1, y_2) \in X \times X$, by Definition 5, Definition 8 and Proposition 1, we get

$$\mu_{A \times B}(x \ast y) = \mu_A(x_1 \ast y_1, x_2 \ast y_2)$$

$$= \mu_{A \times B}(x_1 \ast y_1, x_2 \ast y_1)$$

$$= T(\mu_A(x_1 \ast y_1), \mu_A(x_2 \ast y_1))$$

$$\geq T(T(\mu_A(x_1), \mu_A(y_1)), T(\mu_A(x_2), \mu_A(y_1)))$$

$$= T(T(\mu_A(x_1), \mu_A(x_2)), T(\mu_A(y_1), \mu_A(y_2)))$$

$$= T(\mu_{A \times B}(x_1), \mu_{A \times B}(y_1)),$$

$$\nu_{A \times B}(x \ast y) = \nu_{A \times B}(x_1 \ast y_1, x_2 \ast y_2)$$

$$= \nu_{A \times B}(x_1 \ast y_1, x_2 \ast y_1)$$

$$= S(\nu_A(x_1 \ast y_1), \nu_A(x_2 \ast y_1))$$

$$\leq S(S(\nu_A(x_1), \nu_A(y_1)), S(\nu_A(x_2), \nu_A(y_1)))$$

$$= S(S(\nu_A(x_1), \nu_A(x_2)), S(\nu_A(y_1), \nu_A(y_2)))$$

$$= S(\nu_{A \times B}(x_1), \nu_{A \times B}(y_1)).$$

Hence $A \times B$ is an intuitionistic T-S fuzzy subalgebra of $X \times X$.

**IV. INTUITIONISTIC T-S FUZZY IDEALS IN BCI-ALGEBRAS**

**Definition 9** Let $T$ be a T-norm, $S$ be an S-norm and $T$ and $S$ be dual norm. An intuitionistic fuzzy set $A$ in $X$ is called an intuitionistic T-S fuzzy ideal of $X$ if the following are satisfied:

$$(F_1) \mu_A(0) \geq \mu_A(x),$$

$$(F_2) \mu_A(x) \geq T(\mu_A(x \ast y), \mu_A(y)),$$

$$(F_3) \nu_A(0) \leq \nu_A(x),$$

$$(F_4) \nu_A(x) \leq S(\nu_A(x \ast y), \nu_A(y)),$$

for all $x, y \in X$.

**Proposition 6** Let $A$ be an intuitionistic T-S fuzzy ideal of $X$ and $\mu_A(0) = 1, \nu_A(0) = 0$. If $x \ast y \in X$ holds in $X$, then $\mu_A(x) \geq \mu_A(y), \nu_A(x) \leq \nu_A(y)$.

**Proof.** For all $x, y \in X$, if $x \ast y \leq z$ holds in $X$, then $\mu_A(x) \geq \mu_A(y), \nu_A(x) \leq \nu_A(z)$.

**Proposition 7** Let $A$ be an intuitionistic T-S fuzzy ideal of $X$, $\mu_A(0) = 1$ and $\nu_A(0) = 0$. If the inequality $x \ast y \leq z$ holds in $X$, then $\mu_A(x) \geq T(\mu_A(z), \mu_A(z)), \nu_A(x) \leq S(\nu_A(x), \nu_A(z)).$

**Proof.** For all $x, y, z \in X$, if $x \ast y \leq z$ holds in $X$, then $\mu_A(x) \geq T(\mu_A(x \ast y), \mu_A(y))$.

**Proposition 8** Let $A = \{ (x, \mu_A(x), \nu_A(x)) : x \in X \}$ be an intuitionistic T-S fuzzy ideal of $X$, then $\square A = \{ (x, \mu_A(x), 1 - \mu_A(x)) : x \in X \}$. 


Let $A = \{ (x, 1 - v_\gamma (x), v_\gamma (x)) : x \in X \}$ be an intuitionistic T-S fuzzy ideal of $X$.

**Proof.** Denote $\omega_\gamma (x) = 1 - \mu_\gamma (x)$, for all $x, y \in X$, we have

$$
\omega_\gamma (x) = 1 - \mu_\gamma (x),
$$

$$
\omega_\gamma (x) = 1 - T (\mu_\gamma (x), \mu_\gamma (y))
$$

$$
= T \left( \left( \mu_\gamma (x), \mu_\gamma (y) \right) \right)
$$

$$
= S \left( \left( \mu_\gamma (x), \mu_\gamma (y) \right) \right)
$$

$$
= S \left( 1 - \mu_\gamma (x), 1 - \mu_\gamma (y) \right)
$$

Thus

$$
\Box A = \{ (x, 1 - v_\gamma (x), v_\gamma (x)) : x \in X \}
$$

is an intuitionistic T-S fuzzy ideal of $X$.

**Proposition 9** Let $A$ and $B$ be two intuitionistic T-S fuzzy ideals of $X$, then $A \cap B$ is also an intuitionistic T-S fuzzy ideal of $X$.

**Proof.** For all $x = (x_1, x_2), y = (y_1, y_2) \in X \times X$, by Definition 8, Definition 9 and Proposition 1, we get

$$
\mu_{A \cap B} (x) = T (\mu_A (x), \mu_B (x))
$$

$$
\nu_{A \cap B} (x) = S (\nu_A (x), \nu_B (x))
$$

Hence $A \cap B$ is also an intuitionistic T-S fuzzy ideal of $X$.

**Proposition 10** Let $A$ and $B$ be two intuitionistic T-S fuzzy ideals of $X$, then $A \times B$ is also an intuitionistic T-S fuzzy ideal of $X \times X$.

**Proof.** For all $x = (x, y), y = (y_1, y_2) \in X \times X$, by Definition 8, Definition 9 and Proposition 1, we get

$$
\mu_{A \times B} (x, y) = T (\mu_A (x), \mu_B (y))
$$

$$
\nu_{A \times B} (x, y) = S (\nu_A (x), \nu_B (y))
$$

Hence $A \times B$ is also an intuitionistic T-S fuzzy ideal of $X \times X$.
Since $f: X \to X'$ is an epimorphism from $X$ to $X'$, therefore $F_f(0) = 0'$. Let $x', y' \in X'$, then exist $x, y \in X$, such that $x' = F_f(x), y' = F_f(y)$.

Let $F_f^{-1}(x') = F_f^{-1}(y') = \{x \in X : x \in F_f^{-1}(x), y \in F_f^{-1}(y')\}$, then $F_f^{-1}(x') * F_f^{-1}(y') = F_f^{-1}(x' * y')$, we get $F_f(x * y) = x' * y'$.

Let $0 = F_f(0)$, we have $F_f(0) = F_f(0)$. Therefore, $F_f^{-1}(0) = F_f^{-1}(0)$. Hence $F_f^{-1}(B)$ is an intuitionistic T-S fuzzy ideal of $Y$.

**Proposition 12** If $f: X \to Y$ is an isomorphism mapping from $X$ to $Y$ and $A, B$ are two intuitionistic T-S fuzzy ideals of $X$ and $Y$ respectively then

1. $F_f^{-1}(F_f(A)) = A$.
2. $F_f^{-1}(F_f(B)) = B$.

**Proof.** (1) Since $f$ is an isomorphism mapping from $X$ to $Y$, for all $x \in X$, let $y = f(x)$, then $f^{-1}(y) = \{x\}$ and

$$
\mu_{F_f^{-1}(f^{-1}(y))}(x) = \mu_{F_f^{-1}(A)}(x) = \mu_{F_f^{-1}(f^{-1}(y))}(y) = \mu_{A}(x).
$$

By the similar proof ways, we can have $\nu_{F_f^{-1}(f^{-1}(y))}(x) = \nu_{A}(x)$, for every $x \in X$.

This implies that $F_f^{-1}(F_f(A)) = A$.

**REFERENCES**