Reachable Set Bounding Estimation for Distributed Delay Systems with Disturbances

Li Xu, Shouming Zhong

Abstract—The reachable set bounding estimation for distributed delay systems with disturbances is a new problem. In this paper, we consider this problem subject to not only time varying delay and polytopic uncertainties but also distributed delay systems which is not studied fully until now. We can obtain improved non-ellipsoidal reachable set estimation for neural networks with time-varying delay by the maximal Lyapunov-Krasovskii functional which is constructed as the pointwise maximum of a family of Lyapunov-Krasovskii functionals corresponding to vertices of uncertain polytope. On the other hand, matrix inequalities containing only one scalar and Matlab LMI Toolbox is utilized to give a non-ellipsoidal description of the reachable set. Finally, numerical examples are given to illustrate the existing results.

Keywords—Reachable set, Distributed delay, Lyapunov-Krasovskii function, Polytopic uncertainties.

I. INTRODUCTION

REACHABLE set is a set that bounds all the states starting from the origin by inputs with peak values for a dynamic systems with distributed delays and bounded disturbance inputs. Recently, the problem of finding a smallest bound of reachable set has received considerable attention. It is well known that time delay may result in instability, sustained oscillations, bifurcation or chaos of neural networks which degrades system performance [1]-[4]. So we consider about the reachable set with time delays. It played an important role in peak-to-peak minimization in control theory extensively. Consider the following state-delayed systems with distributed delays and bounded disturbance inputs. Recently, the problem of finding a smallest bound of reachable set has received considerable attention. It is well known that time delay may result in instability, sustained oscillations, bifurcation or chaos of neural networks which degrades system performance [1]-[4]. So we consider about the reachable set with time delays. It played an important role in peak-to-peak minimization in control theory extensively.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following state-delayed systems with distributed delays and bounded disturbance inputs.

\[ \dot{x}(t) = Ax(t) + Bx(t) - h(t) + C \int_{t-h(t)}^{t} x(s) \, ds + D \omega(t) \]

(1)

Where \( x(t) \in \mathbb{R}^n \) is the state vector, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n} \). C, D are known constant matrices with appropriate dimensions belong to a given polytope \( \omega(\omega(\mathbb{R}^l) \) is the disturbance input, the discrete delay \( h(t) \) is a continuous-time differentiable function and the disturbance is a bounded function. We denote \( \Omega = [A, B, C, D] \) and \( \Omega = \sum_{i=1}^{N} \theta_i \Omega_i, \theta_i \geq 0, \sum_{i=1}^{N} \theta_i = 1 \). Here the \( N \) vertices of the polytope are described by \( \Omega_i = \ldots \). by Zhang [9]. Delay-C dependent conditions for estimating the reachable set of the system with distributed delays is derived in the terms of Lyapunov-Krasovskii functional approach and the delay-partitioning technique and so on. We considered the reachable set bound estimation of the time-delay systems with distributed delays in this paper. This problem was investigated by many researchers [10]-[12]. However, it has not been well addressed in many works [13]-[15], which motivates our study. Our main results are different from the works in follow several aspects. First, we select Lyapunov matrices which are more accurate than the vertices of uncertain polytope, which can improve conservatism and get a more accurate description of the reachable set bound. Second, we considered the systems with not only state-delayed but also distributed delays, which is distinguishable from other existing works. Third, we introduce linear matrix inequality techniques as well as convex-hull properties to get a tighter reachable set estimation by splitting integral and bringing conservatism with lower computational complexity to some extent. Finally, numerical examples as well as simulation results are obtained to illustrate the advantages of our treatments.

Notation: Throughout the whole paper \( \mathbb{R}^n \) denotes the \( n \) dimensional Euclidean space and \( \mathbb{R}^{m \times n} \) is the set of all \( m \times n \) real matrices. The superscript \( 'T' \) denotes matrix transposition, and \( I \) and \( 0 \) denote the identity and zero matrix with appropriate dimension. The notation \( P > 0 (P \geq 0) \) means that \( P \) is symmetric and positive definite (positive semi-definite). \( \Theta \) denotes a convex hull. The symmetric terms in a symmetric matrix are denoted by *. Matrix \( \Theta \) if not explicitly stated, are assumed to have compatible dimensions.
that there exists an integer of reachable states with, for any where 

\[ R \in \mathbb{R}^n \]

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\[ P_j > 0 \]

\[ x_0 \in R^n \]

\[ L_i = \{ x \in R^n : x^T P x \leq 1 \} \]

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The reachable set of system (1) is bounded by the intersection of ellipsoids $\varepsilon(P_j, 1)$.

**Proof:** Denoting $T = [T_1, T_2, T_3, T_4, T_5, T_6, T_7]$.

$$
\xi^*(t) = [x^T(t), \dot{x}^T(t), x^T(t-h(t)), x^T(t-h_1(t)), x^T(t-h_2(t))],
$$

$$
- h_2(t)) \cdot \int_{t-d(t)}^t x^T(s) ds, \omega^T(t)]
$$

So we can select the maximal Lyapunov-Krasovskii functional candidate as follows:

$$
V(x) = V_{1, \text{max}}(x) + V_2(x) + V_3(x) + V_1(x)
$$

$$
V_{1,j}(x) = x^T(t) P_j x(t), V_{1, \text{max}}(x) = \max x^T(t) P_j x(t)
$$

$$
V_2(x) = \int_{t-h(t)}^{t} e^{\alpha(s-t)} x^T(s) S x(s) ds + \int_{t-h_1}^{t} e^{\alpha(s-t)} x^T(s) R x(s) ds
$$

$$
V_3(x) = (h_2 - h_1) \int_{t-h_2}^{t-h_1} e^{\alpha(s-t)} x^T(s) E \dot{x}(s) ds
$$

$$
V_4(x) = \int_{t-d(t)}^{t} e^{\alpha(s-t)} \left[ \frac{x(s)}{\dot{x}(s)} \right]^T (P_l H_l) \left[ \frac{x(s)}{\dot{x}(s)} \right] ds \alpha_\theta
$$

To imply the process of proof, we can denote the set:

$$
M_{\text{max}}(x) = \{ j \in \{1, 2, \ldots, M\} : V_{1,j}(x) = V_{1, \text{max}}(x) \}
$$

$$
\gamma_j = \{ x \in \mathbb{R}^n : V_j(x) \geq V_k(x), \forall k \neq j \}
$$

So we can get the inequality $V_{1,j}(x) < V_{1, \text{max}}(x)$, if $j \notin M_{\text{max}}(x)$ as well as $M_{\text{max}}(x) \in [1, m]$ for some integer $m \leq M$. where $m$ is the number of ellipsoids $\varepsilon(P_j, 1)$’s intersected at $x$. So we can get that:

$$
V_{1,j}(x) = V_{1, \text{max}}(x), \text{ for } j \leq m
$$

$$
V_{1,j}(x) < V_{1, \text{max}}(x), \text{ for } j > m
$$

So we can get that:

$$
x^T(P_l - P_k)x \geq 0, \text{ } \forall j \in \{1, 2, 3, \ldots, M\}, k \in [1, M]. \text{ (7)}
$$

Because $x$ is not differentiable everywhere, we should consider two conditions for $x$.

(1) If $x$ is differentiable

$$
x \in \gamma_j \setminus \bigcup_{k \neq j} \gamma_k
$$

$$
V_{1, \text{max}} = \max \{x^T P_j x\}
$$

(2) If $x$ is not differentiable

$$
x \in \bigcap_{j=1}^{i} \bigcup_{k \neq j_1, j_2, \ldots, j_k} \gamma_k, r \geq 2
$$

$$
\nabla_x V_{1, \text{max}}(x) \leq \max_{\xi \in \partial V_{1, \text{max}}(x)} \{ \xi^T \dot{x} \}
$$

$$
= \max_{\xi \in Co(2P_l x, x^T, 1, \ldots, r)} \{ \xi^T \dot{x} \}
$$

$$
\sum_{i=1}^{r} \alpha_r = 1, \alpha_r > 0, P_l > 0, l = (1, 2, 3, \ldots, r)
$$

$$
\dot{V}_2(x) = x^T(t) S x(t) - (1 - h(t)) e^{-\alpha h(t)} x^T(t - h(t)) S x(t) - h_1(t) + x^T(t) Q x(t) - e^{-\alpha h_1(t)} x^T(t - h_1(t)) Q x(t) - \alpha V_2
$$

$$
\leq x^T(t) (S + Q + R) x(t) - (1 - h(t)) e^{-\alpha h_2 x^T(t - h(t))} x^T(t - h(t)) S x(t) - \alpha h(t) e^{-\alpha h_1(t)} Q x(t) - \alpha V_2
$$

$$
\leq x^T(t) (S + Q + R) x(t) - e^{-\alpha h_2 x^T(t - h(t))} x^T(t - h(t)) S x(t) - \alpha h(t) e^{-\alpha h_1(t)} Q x(t) - \alpha V_2
$$

$$
\dot{V}_3(x) = (h_2 - h_1) \int_{t-h_2}^{t-h_1} x^T(s) E \dot{x}(s) ds - \alpha V_3
$$

$$
\dot{V}_4(x) = d(t) \left[ x(t) \right]^T \left[ \frac{x(t)}{\dot{x}(t)} \right] (P_l H_l) \left[ \frac{x(t)}{\dot{x}(t)} \right] ds - \alpha V_4
$$

$$
\leq d_2 \left[ x(t) \right]^T \left[ \frac{x(t)}{\dot{x}(t)} \right] (P_l H_l) \left[ \frac{x(t)}{\dot{x}(t)} \right] - \frac{e^{-\alpha d(t)}}{d_2}
$$
we can get the following equation holds by the system(1):

\[
2\xi T(t)T^T[-\dot{x}(t) + Ax(t) + Bx(t-h(t))] + C \int_{t-d(t)}^t x(s)ds + D \omega(t) = 0 \\
2(x^T(t)T_1^2 + x^T(t-T_2^2) + x^T(t-h(t))T_3^2 + x^T(t-h_1)T_4^2) \\
+ x^T(t-h_2)T_5^2 + \frac{1}{d_1} \int_{t-d(t)}^t x^T(s)dsT_6^2 + \omega(t)(t)T_7^2 \\
+ [-\dot{x}(t) + Ax(t) + Bx(t-h(t))] + C \int_{t-d(t)}^t x(s)ds + D \omega(t) = 0
\]

(1) If \(x\) is differentiable
\[x \in \gamma_j \setminus \bigcup_{k \neq j} \gamma_k\]

We can get following inequality by using the convex property of the polytope and S procedure.

\[
\Pi \triangleq \frac{dV(x)}{dx} + \alpha V(x) - \frac{\alpha}{\omega_m} \omega^T(t)\omega(t) \\
\leq \xi^T(t)\Omega_1 \xi(t) = \xi^T(t)\Sigma_{i=1}^N 1_{\theta_i} \Omega_1 \xi(t) - \Sigma_{i=1}^M 1_{\theta_i} \beta_j x^T(t) \left(P_j - P_h\right) x(t) \leq 0
\]

Where
\[
\Omega_1 = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & A^T T_4 & A^T T_5 & \Omega_{16} & \Omega_{17} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} & -T_4 & -T_5 & \Omega_{26} & \Omega_{27} \\
\Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} & \Omega_{35} & \Omega_{36} & \Omega_{37} \\
\Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} & \Omega_{45} & \Omega_{46} & \Omega_{47} \\
\Omega_{51} & \Omega_{52} & \Omega_{53} & \Omega_{54} & \Omega_{55} & \Omega_{56} & \Omega_{57} \\
\Omega_{61} & \Omega_{62} & \Omega_{63} & \Omega_{64} & \Omega_{65} & \Omega_{66} & \Omega_{67} \\
\Omega_{71} & \Omega_{72} & \Omega_{73} & \Omega_{74} & \Omega_{75} & \Omega_{76} & \Omega_{77}
\end{bmatrix} < 0
\]

\[
\tilde{\Omega}_{11} = S + Q + R + \alpha P_j + d_2 F_1 - \frac{e^{-\alpha d_2}}{d_2} Q_1 + T_1^2 A \\
+ A^T T_1 + \sum_{k=1}^M 1_{\theta_k} \beta_j (P_j - P_h) \\
\tilde{\Omega}_{12} = P_j + d_2 H_1 - T_1^2 + A^T T_3 \\
\tilde{\Omega}_{13} = \frac{e^{-\alpha d_2}}{d_2} Q_1 + T_1^2 B + A^T T_3 \\
\tilde{\Omega}_{16} = \frac{e^{-\alpha d_2}}{d_2} H_1 + T_1^2 C + A^T T_6 \\
\tilde{\Omega}_{17} = T_1^2 D + A^T T_7
\]

\[
\tilde{\Omega}_{23} = T_2^2 B - T_3 \\
\tilde{\Omega}_{26} = T_2^2 C - T_6 \\
\tilde{\Omega}_{27} = T_2^2 D - T_7 \\
\tilde{\Omega}_{33} = \frac{e^{-\alpha d_2}}{d_2} Q_1 - (1-h) \frac{e^{-\alpha b_2}}{h_2} S + \frac{e^{-\alpha b_2}}{h_2} (-2E + G + G^T) + T_3^2 B + B^T T_3 \\
\tilde{\Omega}_{34} = \frac{e^{-\alpha b_2}}{d_2} (E^T - G) + B^T T_4 \\
\tilde{\Omega}_{35} = \frac{e^{-\alpha b_2}}{d_2} (E^T + G^T) + B^T T_5 \\
\tilde{\Omega}_{36} = \frac{e^{-\alpha b_2}}{d_2} h_1 + T_6^2 C + B^T T_6 \\
\tilde{\Omega}_{37} = T_6^2 D - B^T T_7 \\
\tilde{\Omega}_{66} = \frac{e^{-\alpha d_2}}{d_2} F_1 + T_7^2 C \\
\tilde{\Omega}_{67} = T_7^2 D + C^T T_7 \\
\tilde{\Omega}_{77} = T_7^2 D - \frac{\alpha}{\omega_m} I
\]

We can get the results:
\[
\Pi - \Sigma_{i=1}^M 1_{\theta_i} \beta_j x^T(t)(P_j - P_h) x(t) \leq 0 \\
\Pi \leq \Sigma_{i=1}^M 1_{\theta_i} \beta_j x^T(t)(P_j - P_h) x(t)
\]

This implies that \(V_{max} \leq V(x) \leq 1\) by Lemma 3.

(2) If \(x\) is not differentiable,
\[x \in \bigcap_{j=1}^{i_r} \gamma_j \setminus \bigcup_{k \neq i_r} \gamma_k, \gamma_r \geq 2\]

We can get following inequality by using the definition of the subdifferential and S procedure.

\[
\Pi \triangleq \frac{dV(x)}{dx} + \alpha V(x) - \frac{\alpha}{\omega_m} \omega^T(t)\omega(t) \\
\leq 2x^T P_j \dot{x} + \alpha x^T(t)P_j x(t) + \dot{\mathcal{V}}_2(t) + \dot{\mathcal{V}}_3(t) + \dot{\mathcal{V}}_4(t) \\
- \frac{\alpha}{\omega_m} \omega^T(t)\omega(t) \\
\leq \Sigma_{i=1}^M 1_{\theta_i} \beta_j x^T(t)(P_j - P_h) x(t) \leq 0
\]

Therefore, we can get \(V_{max} \leq V(x) \leq 1\) by Lemma 3. This completes the proof of Theorem 1.

Remark 1. In this paper, we considered the reachable set bounding estimation for delay systems with disturbances. However, we considered just the systems before: \(\dot{x}(t) = Ax(t) + Bz(t-h(t)) + \dot{\mathcal{W}}(t)\) of the systems:
\[
\dot{x}(t) = Ax(t) + C \int_{t-d(t)}^t x(s)ds + D \omega(t)
\]
What is more, \(-\frac{h_2 - h_1}{h_2} \int_{t-h_2}^{t-h_1} e^{\alpha(x-s)} \dot{x}^T(s)E\dot{x}(s)ds\) is enlarged by \(-h_2 - h_1 \int_{t-h_2}^{t-h_1} e^{\alpha(x-s)} \dot{x}^T(s)E\dot{x}(s)ds\), where an other term \(-h_2 - h_1 \int_{t-h_2}^{t-h_1} e^{\alpha(x-s)} \dot{x}^T(s)E\dot{x}(s)ds\) is ignored [16]- [18]. The useful term is explored in this paper. So we can reduce the conservativeness for our results.

Remark 2. In [3], we can pay attation that the Lemma 1 of [3], Zuo ignored the term \(-\frac{d(t)}{d(t)}x^T(t-d(t)) - x^T(t-h(t))R(x(t-d(t))-x(t-h(t)))\) and the term \(-\frac{d(t)}{d(t)}(x^T(t-d(t)))R(x(t-d(t))-x^T(t-d(t)))\). In this paper, we considered the two terms above and can get
the matrix $\begin{bmatrix} E & G \end{bmatrix}$ better than the matrix $\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$, by using inequality. So it can be theoretically proven that it has less conservativeness and can be used more further.

**Remark 3.** We select free-weighting matrices $T = [T_1, T_2, T_3, T_4, T_5, T_6, T_7]$ and can decouple between the system matrices and the Lyapnov matrices by using (11).

We can make a treating which is simple optimization problem formulation to look for the accurate bound for the reachable system matrices and the Lyapnov matrices by using (11).

We can write the following optimization problem:

Min:

\[
\sigma I \leq \sigma I - \sigma I P_\sigma \geq 0.
\]

**IV. NUMERICAL EXAMPLES**

In this section, we provide the simulation of examples to illustrate the effectiveness of our method. We can select following parameters:

\[
A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.7 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & 0 \\ -1 & -0.9 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1.1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1.1 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} -1 & -1 \\ 0 & 0.9 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.
\]

The resulting $\delta$s are listed in Table I for different values of $d$. By solving the optimization problem (7), we can get a tighter bounds than the approach derived in [4].

**REFERENCES**


