A PROOF FOR GOLDBACH’S CONJECTURE

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Abstract

In 1742, Goldbach claimed that each even number can be shown by two primes. In 1937, VinogradOV of Russian Mathematician proved that each odd large number can be shown by three primes. In 1930, LevSchnirelmann proved that each natural number can be shown by $M$-primes
In 1973, Chen Jingrun proved that each odd number can be shown by one prime plus a number that has maximum two primes. In this article, we state one proof for Goldbach’s conjecture.
1. Introduction

Bertrand’s postulate states for every positive integer $n$, there is always at least one prime $p$, such that $n < p < 2n$. This was first proved by Chebyshev in 1850, which is why the postulate is also called the Bertrand-Chebyshev theorem.
Legendre’s conjecture states that there is a prime between 

\[ n^2 \text{ and } (n + 1)^2 \] 
for every positive integer \( n \), which is one of the four Landau problems.

The rest of these four basic problems are:

(i) Twin prime conjecture: There are infinitely many primes \( p \) such that \( p + 2 \) is a prime.
(ii) Goldbach’s conjecture: Every even integer

\[ n > 2 \text{ can be written as the sum of two primes.} \]

(iii) Are there infinitely many primes \( p \) such that \( p - 1 \) is a perfect square?
Theorem. Every even large positive integer can be written of two primes.

Before solving this theorem, we state some equations as follow:

\[ N - p_1 = m_1 \]
\[ N - p_2 = m_2 \]
\[ \ldots \]
\[ N - p_r = m_r \]
That $p_1 = 3, p_2 = 5, ...$

Notice that $p_r$ is a largest prime such that $\frac{N^2}{3} < p_r < 2\frac{N^2}{3}$

We know there is a prime in this interval, $N$ is a large even number and $m_1, m_2, ... m_r$ are composite odd numbers, otherwise our theorem is correct.

We prove this theorem by induction and we assume that all $m_1, m_2, ... m_r$ composite and we reach to a contradiction.

To proceed to this proof, firstly, we use the following lemmas:
2. Lemmas

In this section, we present several lemmas, which are used in the proof of our main theorem.

Before state the below lemmas, we write $m_1 \cdot m_2 \ldots m_r < m_1 \cdot a_1 \cdot a_2 \ldots a_{r-1}$ that $a_1 \cdot a_2 \ldots a_{r-1}$ are composite odd numbers, less than $m_1$, and may be equal or larger than $m_2, m_3, \ldots m_r$, for simplicity we denote $m_1 = m_0$.

**Lemma 2.1.** If the number of composite odd numbers to be $r$ so the number of total odd numbers (composite and prime) are at most $9r/4$. 

Lemma 2.2. If $N^{2/3} \leq pr \leq 2N^{2/3}$, then for a large $r$, $N$, we have $r \geq N^{20/33}$

Lemma 2.3. All prime factors $q$, where $3 \leq q \leq N$ appeared in Numbers $a_0, a_1, a_2, \ldots a_{r-1}$

Lemma 2.4. Let $l$ denote the number of $3 \leq q \leq a_0$, are in odd composite numbers. $a_0, a_1, a_2, \ldots a_{r-1}$ so $l \leq 9r/4$ for all $3 \leq q \leq a_0$

Lemma 2.5. Let $p$ is prime and $f$ denote the number of $p > \sqrt{a_i}$, in which $0$ $1 \leq i \leq r - 1$ are in $a_0, a_1, a_2, \ldots a_{r-1}$, i.e $a_i = tp$, these numbers are
odd composite numbers.

So the number of such $p$, are:

$$ f \leq \frac{9r/4}{9} $$

Or

$$ f \leq \frac{9r/4(1-\frac{1}{9})}{3\times5} $$

Or

$$ f \leq \frac{9r/4(1-\frac{1}{9}-1/15)}{3\times7} $$
we continue this method to reach $1 - 1/9 - 1/15 - 1/21 - ... - 1/267 = a$lmost 0.56, because we have $r$ odd composite numbers.

Proof. If $p > \sqrt{a_i}$, in which $0 \leq i \leq r - 1$, that $a_i = qp$, so $a_{r-1}/q < p < a_0/q$, in which $3 \leq q \leq \sqrt{a_0}$. Since the distance of between two odd numbers should be 2, so $\frac{a_{r-1}}{q} - 2 \leq p - 2 = p' \leq \frac{a_0}{q} - 2$ that $p'$ is prime but in $\frac{a_{r-1}}{q} - 4 \leq p - 4 = w \leq \frac{a_0}{q} - 4$, $w$ is not prime, so if $q = 3$, the number of such $p$ is:

$$f \leq \frac{9r/2}{2 \times 3} \times \frac{2}{3} \times \frac{1}{2} = \frac{9r/4}{9}$$
But since $p > \sqrt{a_0}$, only one $p > \sqrt{a_0}$, in which $0 < i < r - 1$, could be in $a_0, a_1, a_2, \ldots a_{r-1}$, so for $q = 5$,

$$f \leq \frac{9r/4(1 - \frac{1}{9})}{3 \times 5}$$

For $q=7$, 

$$f \leq \frac{9r/4(1-\frac{1}{9}-\frac{1}{15})}{3 \times 7}$$

we continue this method to reach $1 - 1/9 - 1/15 - 1/21 - \ldots - 1/267 = \text{almost } 0.56$.

Note. We have only $9r/4$ odd numbers, since we say about $p > \sqrt{a_0}$ (this is new idea), not old idea i.e $q \leq \sqrt{a_0}$, for $q = 3$, we have $(9r/4)/9$
such $p > \sqrt{a_0}$, since we have only one such $p > \sqrt{a_0}$, if we have two such primes, i.e., $p_1 p_2q > a_0$ and this is contradiction, so for $q = 5$ ($9r/4$) numbers changed to $(9r/4) - (9r/4)/9$, for $q = 7$, these numbers changed to $9r/4 - (9r/4)/9 - (9r/4)/15$, we continue this method to reach $9r/4 - (9r/4)9 - (9r/4)15 - ... - (9r/4)/3 \times 89 = \text{almost 0.56}$, since we have only $r$ composite numbers.

**Lemma 2.6:** we always have:

$$\log(N - 3) \ldots (N - p_r) > r \log(N - p_r).$$
The Proof of Main Theorem

**Theorem.** *Every even large positive integer can be written as the sum of two primes.*

**Proof.** Let all even numbers smaller than $N$ be represented by two primes by induction, and we assume that all $a_0, a_1, a_2, ... a_{r-1}$ are composite odd numbers, and we reach to a contradiction. According to (Hardy and Wright [3]), there is a prime factor like $q$ that for any composite odd numbers $a_0, a_1, a_2, ... a_{r-1}, q \leq \sqrt{a_i}$ in which $0 \leq i \leq r - 1$, Now, we use the above results to reach a contradiction. According to

Lemma 2.6, $\log(N - 3) \ldots (N - p_r) > r \log(N - p_r)$ and according to lemma 2.5:

$$(N - 3) \ldots (N - p_r) \leq 3^{\frac{9r}{4}} \times \ldots \times 89^{\frac{9r}{4}} \times \frac{N}{3} \times \frac{N}{5} \times \ldots$$
We continue to reach $1/9_1 1/15_1 1/21_1 \ldots_1 1/267 \approx 0.56$. Hence, we have:

$$r \log(N - p_r) < \frac{9r}{4} \sum_{3 \leq q \leq 89} \frac{\log q}{q} + \frac{9r}{4} \left( \frac{1}{9} + \frac{(1-\frac{1}{9})}{3 \times 5} + \frac{(1-\frac{1}{9}-\frac{1}{15})}{3 \times 7} + \ldots + \right)$$

u) $\log N$

So by refer to (Hardy and Wright [3]), $\sum_{3 \leq q \leq w} \frac{\log q}{q} < \log w + c$, that $c$ is positive constant number, so

$$\log(N - p_r) < \frac{9}{4} \log 89 + \frac{9}{4} c + 0.85 \log N$$

Then for a large $N$,

$$N < N^{0.9} + p_r$$

But by our assume $p_r \leq 2N^{\frac{2}{3}}$, so we have:

$$N < N^{0.9} + 2N^{\frac{2}{3}}$$
This is contradiction for Large N

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