Uniformly Strong Persistence for a Predator-Prey Model with Modified Leslie-Gower and Holling-Type II Schemes

Changjin Xu, Yuseun Wu

Abstract—In this paper, a asymptotically periodic predator-prey model with Modified Leslie-Gower and Holling-Type II schemes is investigated. Some sufficient conditions for the uniformly strong persistence of the system are established. Our result is an important complementarity to the earlier results.

Keywords—Predator-prey model, uniformly strong persistence, asymptotically periodic, Holling-type II.

I. INTRODUCTION

It is well known that the dynamical behavior of predator-prey systems is a form of very common biological inter action in the natural word. This topic has attracted a lot of attention and many good results have already been reported. For example, Chen and Chen [1] studied the linear stability of trivial periodic solution and semi-trivial periodic solutions of a periodic predator-prey system with distributed time delays and impulsive effect. Mukherjee [2] made a discussion on the uniform persistence in a generalized prey-predator system with parasitic infection. Chen [3] gave a theoretical study on the almost periodic solution of the non-autonomous two-species competitive model with stage structure. Sen et al. [4] analyzed the bifurcation behavior of a ratio-dependent prey-predator model with the Allee effect. Agiza et al. [5] investigated the chaotic phenomena of a discrete prey-predator model with Holling type II. Aggelis et al. [6] considered the coexistence of both prey and predator populations of a prey-predator model. Nindjin and Aziz-Alaoui [7] focused on the persistence and global stability in a delayed Leslie-Gower type species food chain. Ko and Ryu [8] discussed the coexistence states of a nonlinear Lotka-Volterra type predator-prey model with cross-diffusion. Fazly and Hesaaraki [9] dealt with periodic solutions of a predator-prey system with monotone functional responses. One can see [10-52] etc. For more related studies. However, the research work on asymptotically periodic predator-prey model is very few at present.

The so-called asymptotically periodic function is the function \( a(t) \) which can be expressed by the form \( a(t) = a(t) + \tilde{a}(t) \), where \( a(t) \) is a periodic function and \( \tilde{a}(t) \) satisfies \( \lim_{t \to +\infty} \tilde{a}(t) = 0 \).

In 2003, Aziz-Alaoui and Okiye [53] investigated the stability and bifurcation of the following predator-prey model with time delay

\[
\begin{align*}
\frac{dx}{dt} &= x(t) \left[ a - bx(t) - \frac{cy(t)}{x(t) + k_1} \right], \\
\frac{dy}{dt} &= y(t) \left[ d - \frac{e(t)y(t)}{x(t) + k_2} \right],
\end{align*}
\]

with initial conditions \( x(0) \geq 0, y(0) \geq 0 \), where \( x(t) \) denotes the densities of prey, at time \( t \); \( y(t) \) denotes the densities of the predator at time \( t \); \( a, b, c, d, e, k_1, k_2 \) are all positive constants.

II. UNIFORMLY STRONG PERSISTENCE

In this section, we shall present some result about the uniformly strong persistence of system (2). For convenience and simplicity in the following discussion, we introduce the notations, definitions and Lemmas. Let

\[
0 < f^i = \lim_{t \to +\infty} \inf_{t \to +\infty} f(t) \leq \lim_{t \to +\infty} \sup_{t \to +\infty} f(t) = f^u < +\infty.
\]
In view of the definitions of lower limit and upper limit, it follows that for any \(\varepsilon > 0\), there exists \(T > 0\) such that
\[
f^{t} - \varepsilon \leq f(t) \leq f^{u} + \varepsilon, \quad t \geq T.
\]

**Definition 1.** The system (2) is said to be strong persistence, if every solution \(x(t)\) of system (2) satisfies
\[
0 < \lim_{t \to +\infty} \inf x(t) \leq \lim_{t \to +\infty} \sup x(t) \leq \delta < +\infty.
\]

**Lemma 1.** Both the positive and nonnegative cones of \(R^2\) are invariant with respect to system (2).

It follows from Lemma 1 that any solution of system (2) with a nonnegative initial condition remains nonnegative.

**Lemma 2.** If \(a > 0, b > 0\) and \(\dot{x}(t) \geq \langle \infty \rangle x(t)(b - ax^\alpha(t))\), where \(\alpha\) is a positive constant, when \(t \geq 0\) and \(x(0) > 0\), we have
\[
x(t) \geq \left(\frac{b}{a}\right) \frac{\alpha}{\alpha - 1} \left[1 + \left(\frac{bx^{\alpha - 1}(0)}{a} - 1\right) e^{-\alpha t}\right]^{\frac{1}{\alpha}}.
\]

In the following, we will ready to state our result.

**Theorem 1.** Let \(\theta_2\) be defined by (9). Assume that the condition (H) and \(a^1k_1^2 > c^\theta_2\) hold, then system (2) is uniformly strong persistence.

**Proof** It follows from (3) that for any \(\varepsilon > 0\), there exists \(T_1 > 0\) such that for \(t > T_1\),
\[
\begin{align*}
d^f - \varepsilon &\leq a^f(t) \leq a^u + \varepsilon, -\varepsilon < \tilde{a}(t) < \varepsilon, \\
b^f - \varepsilon &\leq b^f(t) \leq b^u + \varepsilon, -\varepsilon < \tilde{b}(t) < \varepsilon, \\
e^f - \varepsilon &\leq e(t) \leq e^u + \varepsilon, -\varepsilon < \tilde{e}(t) < \varepsilon, \\
d^l - \varepsilon &\leq d(t) \leq d^u + \varepsilon, -\varepsilon < \tilde{d}(t) < \varepsilon, \\
k^1 - \varepsilon &\leq k_1(t) \leq k^1 + \varepsilon, -\varepsilon < \tilde{k}_1(t) < \varepsilon, \\
k^2 - \varepsilon &\leq k_2(t) \leq k^2 + \varepsilon, -\varepsilon < \tilde{k}_2(t) < \varepsilon.
\end{align*}
\]

Substituting (4) into the first equation of system (2), we have
\[
\frac{dx}{dt} = x(t) \left[ a(t) + \tilde{a}(t) - (b(t) + \tilde{b}(t))x(t) \right] - \frac{(c(t) + \tilde{c}(t))y(t)}{x(t) + k_1(t) + k_2(t)}
\leq x(t) \left[ a(t) + \tilde{a}(t) - (b(t) + \tilde{b}(t))x(t) \right]
\leq x(t) \left[ (a^u + 2\varepsilon) - (b^u - 2\varepsilon)x(t) \right].
\]

By Lemma 2, we get
\[
\lim_{t \to +\infty} \sup x(t) \leq \frac{a^u}{b^u} := \theta_1.
\]

Then for any \(\varepsilon > 0\), there exists \(T_2 > T_1 > 0\) such that
\[
x(t) \leq \theta_1 + \varepsilon, \quad t \geq T_2.
\]

Similarly, from (3) and the second equation of system (2), we obtain that for any \(\varepsilon > 0\), there exists \(T_3 > T_2 > 0\) such that
\[
\dot{y}(t) = y(t) \left[ d(t) + \tilde{d}(t) - \frac{(e(t) + \tilde{e}(t))y(t)}{x(t) + k_2(t) + k_2(t)} \right]
\leq y(t) \left[ (d^u - 2\varepsilon) - \frac{(e^u - 2\varepsilon)y(t)}{\theta_1 + \varepsilon + k^2_2 + 2\varepsilon} \right].
\]

In view of Lemma 2, we derive
\[
\lim_{t \to +\infty} \sup y(t) \leq \frac{d^u(\theta_1 + k^2_2)}{e^u} := \theta_2.
\]

Thus for any \(\varepsilon > 0\), there exists \(T_4 > T_3 > 0\) such that
\[
y(t) \leq \theta_2 + \varepsilon, \quad t \geq T_4.
\]

By (7), (10) and the first equation of system (2), we obtain that for any \(\varepsilon > 0\), there exists \(T_5 > T_4 > 0\) such that
\[
\frac{dx}{dt} = x(t) \left[ a(t) + \tilde{a}(t) - (b(t) + \tilde{b}(t))x(t) \right]
\leq x(t) \left[ (a^u - 2\varepsilon) - (b^u + 2\varepsilon)x(t) \right]
\leq \frac{(c^u + 2\varepsilon)(\theta_2 + \varepsilon)}{k^2_1 - 2\varepsilon}.
\]

Using Lemma 2 again, we have
\[
\lim_{t \to +\infty} \inf x(t) \geq \frac{a^1k_1^2 - c^\theta_2}{k^2_1b^u} := \delta_1.
\]

Thus for any \(\varepsilon > 0\), there exists \(T_6 > T_5 > 0\) such that
\[
x(t) \geq \delta_1 - \varepsilon.
\]

According (7), (10) and the second equation of system (2), we obtain that for any \(\varepsilon > 0\), there exists \(T_7 > T_6 > 0\) such that
\[
\dot{y}(t) = y(t) \left[ d(t) + \tilde{d}(t) - \frac{(e(t) + \tilde{e}(t))y(t)}{x(t) + k_2(t) + k_2(t)} \right]
\geq y(t) \left[ (d^u + 2\varepsilon) - \frac{e^u + 2\varepsilon}{k^2_2 - 2\varepsilon}y(t) \right].
\]

Using Lemma 2 again, we have
\[
\lim_{t \to +\infty} \inf y(t) \geq \frac{d^u}{k^2_2} := \delta_2.
\]

Thus we complete the proof of Theorem 1.

**III. Conclusions**

In this paper, we have investigated a asymptotically periodic predator-prey model with modified Leslie-gower and Holling-type II schemes. A set of sufficient conditions for the uniformly strong persistence of the system are derived. It is shown that under some suitable conditions, the asymptotically periodic predator-prey model with modified Leslie-Gower and Holling-type II schemes is uniformly strong persistence. Our results obtained in this paper complement the earlier results.

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Changjin Xu graduated from Huaibua University, China, in 1994. He received the M. S. degree from Kunming University of Science and Technology in 2004 and the Ph.D. degree from Central South University, China, in 2010. He is currently a professor at the Guizhou Key Laboratory of Economics System Simulation at Guizhou University of Finance and Economics. He has published about 100 refereed journal papers. He is a Reviewer of Mathematical Reviews and Zentralbatt-Math. His research interests include nonlinear systems, neural networks, stability and bifurcation theory.

Yusen Wu graduated from Liaocheng University, P.R. China, in 2004. He received the M. S. degree from Central South University, P.R. China in 2007 and the Ph.D. degree from Central South University, P.R. China, in 2010. He is currently an associate professor at School of Mathematics and Statistics of Henan University of Science and Technology. He has published about 30 refereed journal papers. He is a Reviewer of Mathematical Reviews. His research interests include the qualitative theory of ordinary differential equation and computer symbol calculation.