On CR-Structure and F-Structure Satisfying Polynomial Equation

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Abstract—The purpose of this paper is to show a relation between CR structure and F-structure satisfying polynomial equation. In this paper, we have checked the significance of CR structure and F-structure on integrability conditions and Nijenhuis tensor. It was proved that all the properties of integrability conditions and Nijenhuis tensor are satisfied by CR structures and F-structure satisfying polynomial equation.

Keywords—CR-submanifolds, CR-structure, Integrability condition & Nijenhuis tensor.

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I. INTRODUCTION

The study of F structure and CR structure is done by many mathematicians. In this paper the study of these structures are considered with polynomial equations, the study of integrability and Nijenhuis tensor is also extended to polynomial equation. Yano [1] initiated the study of F structure, Nikie [8] and Das [9] further studied the properties of F structure.

Let \( F \) be a non zero tensor field of type \((1,1)\) and of class \(C^\infty\) dimensional manifold \( M \) such that

\[
a_n F^n + a_{n-1} F^{n-1} + \cdots a_2 F^2 + a_1 F = 0
\]

where \( n \) is a fixed positive integer greater than or equal to 1. Such a structure on \( M \) is called an F-structure. If the rank of \( F \) is constant and \( r = r(F) \), then \( M \) is called an F structure manifold of degree \( n \).

Let us define the operator on \( M \) as:

\[
l = (\alpha_1 a_{n-1} F^{n-1} + \cdots a_2 F^2 + a_1 F^1) \frac{\partial}{\partial x_a}
\]

\[
m = I + (\alpha_1 a_{n-1} F^{n-1} + \cdots a_2 F^2 + a_1 F^1)
\]

where I denotes the identify operator on \( M \).

Theorem 1. Let \( M \) be an \( F(a_n, a_{n-1}, \ldots, a_1) \) structure manifold satisfying (1) then

a) \( l + m = I \)

b) \( l^2 = I \)

c) \( m^2 = m \)

d) \( l.m = 0 \)

Proof.

\[
l + m = I
\]

\[
l + m = -\left( a_n F^n + a_{n-1} F^{n-1} + \cdots a_2 F^2 + a_1 F^1 \right) + I
\]

\[
\Rightarrow l + m = I
\]

(4)

b) \( l^2 = I \)

\[
l^2 = -\left( a_n F^n + a_{n-1} F^{n-1} + \cdots a_2 F^2 + a_1 F^1 \right) + I
\]

\[
\Rightarrow l^2 = I
\]

(5)

m^2 = m

\[
m^2 = I + I = l + l
\]

(6)

\[
l.m = 0.
\]

(7)

For \( F \neq 0 \) satisfying (1) there exist complimentary distributions \( D_l \) & \( D_m \) corresponding to the projection operator \( l \) & \( m \) respectively. If \( \text{Rank } F = \text{constant} \) and \( r = r(F) \), then \( \text{dim } D_l = r \) and \( D_m = n - r \).

Theorem 2. We have-

a) \( (I) \text{ If } F = Fl \)

b) \( (l) (a_n F^n + a_{n-1} F^{n-1} + \cdots a_2 F^2 + a_1 F^1) = 0 \)

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(II) \[
\left( \frac{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2}{a_1} \right) * \mathbf{f} = - \mathbf{f}
\]

**Proof.**

(a) \( IF = Fl = F \)
\[
IF = \left( \frac{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2}{a_1} \right) * F = \left[ F + \left( \frac{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2}{a_1} \right) \right] F
\]
So \( IF = Fl = F \) \hspace{1cm} (8)

(b) \( mF = Fm = 0 \)
\[
mF = I \left( \frac{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2}{a_1} \right) m = 0
\]
\[
mF = \left[ I + \left( \frac{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2}{a_1} \right) \right] F + (-F) = 0
\]
So \( mF = Fm = 0 \) \hspace{1cm} (9)

Thus, \( F \) acts on \( D \) as an almost complex structure and on \( D_m \) as a null operator.

II. **NIJENHUIS TENSOR**

The Nijenhus tensor \( N(X,Y) \) of \( F \) satisfying (1) in \( M \) is expressed as follows for every vector field \( X, Y \) on \( M \).

\[
N(X,Y) = [FX, FY] - [X, FY] - [X, FY] + [X, FY]
\]

We state the following theorem without proof

**Theorem 3.** A necessary & sufficient condition for the \( f \)-structure to be integrable is that

\[
N(X,Y) = 0
\]

\hspace{1cm} (12)

III. **LIE BRACKET**

If \( X \) & \( Y \) are two vector fields in \( M \) then their lie bracket \( [X, Y] \) is defined by

\[
[X, Y] = XY - YX
\]

IV. **CR-STRUCTURE**

A study of differential geometry of a CR submanifold has been initiated in [4]-[7]. Results on general theory of Cauchy Riemann manifolds have been obtained by [2].

Let \( M \) be a differentiable manifold and \( T_c (M) \) be its complex field on tangent bundle \( M \). A CR-Structure on \( M \) is a complex sub bundle \( H \) of \( T_c (M) \) such that \( H_p \cap H_p^* = 0 \) & \( H \) is involutive i.e. for complex vector field \( Y \) in \( H \), \( [X, Y] \) in \( H \). In this case we say \( M \) is a CR-manifold.

Let \( F(a_n, a_{n-1}, \ldots, a_i) \) be an integrable structure satisfying (1) of rank \( r = 2m \) on \( M \). We define complex sub bundle \( H \) of \( T_c (M) \) by

\[
H_p = \{ X - \sqrt{-1} FX, X \in \chi (Dl) \}
\]

where \( \chi (Dl) \) is the \( F(Dl_m) \) module for all differentiable sections of \( Dl \). The \( \text{Re}(H) = D_l \) & \( H_p \cap H_p^* = 0 \), where \( H_p \) denotes the complex conjugate. Intrigability conditions on such submanifolds have been investigated by [4].

**Theorem 4.** If \( P & Q \) are two elements of \( H \) then the following relation holds

\[
[P, Q] = [X, Y] - [FX, FY] - \sqrt{-1} [FX, FY] - \sqrt{-1} [FX, FY]
\]

**Proof.** Let us define

\[
P = X - \sqrt{-1} FX
\]
\[
Q = Y - \sqrt{-1} FY
\]

then by direct calculation & on simplifying, we obtain

\[
[P, Q] = [X, Y] - [FX, FY] - \sqrt{-1} [FX, FY] - \sqrt{-1} [FX, FY]
\]

**Theorem 5.** If \( F(a_n, a_{n-1}, \ldots, a_i) \) structure satisfying (1) is integrable then we have

\[
\left( a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2 \right) \left( [FX*FY] + [X, FY] \right) = l
\]

\[
{[FX, Y] + [X, FY]}
\]

**Proof.** From (12) we have,

\[
\]

Since \( N(X,Y) = 0 \) we obtain

\[
\]

Operating \[
\left( a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2 \right) [FX, FY] + F^2 [X, Y]
\]

\[
\left( a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2 \right) F(X, FY) + [X, FY]
\]

\[
\left( a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2 \right) [FX, Y] + [X, FY]
\]

\[
\left( a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2 \right) [FX, Y] + [X, FY]
\]

\[
\left( a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2 \right) [FX, Y] + [X, FY]
\]

This proves the above theorem.

**Theorem 6.** The following identities hold

\[
mN(X,Y) = m[FX, FY]
\]
\[
mN\left( a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2 \right) X, Y
\]

\[
m\left( a_n p^n + a_{n-1} p^{n-1} + \ldots + a_3 p^3 + a_2 p^2 \right) F(X, Y)
\]

**Proof.**
a) \( mN(X,Y) = m\{ [FX,FY] + F^2 [X,Y] - F[FX,Y] - F[X,FY] \} \)

\[

\[\Rightarrow mN(X,Y) = m[F,F] \] (16)

b) \( mN\left( \frac{a_n F^{n+1} + a_{n+1} F^{n} + \cdots + a_1 F^2 + a_2 F}{a_1} \right) X, Y = m \)

\[
mN\left( \frac{a_n F^{n+1} + a_{n+1} F^{n} + \cdots + a_1 F^2 + a_2 F}{a_1} \right) X, Y = m \]

By the equation \( mF = 0 = Fm \)

Theorem 7. For any two vector field \( X \) & \( Y \), the following condition are equivalent –

a) \( m (X,Y) = 0 \)

b) \( m[X,FY] = 0 \)

c) \( mN\left( \frac{a_n F^{n+1} + a_{n+1} F^{n} + \cdots + a_1 F^2 + a_2 F}{a_1} \right) X, Y = 0 \)

d) \( m\left( \frac{a_n F^{n+1} + a_{n+1} F^{n} + \cdots + a_1 F^2 + a_2 F}{a_1} \right) X, Y = 0 \)

e) \( m\left( \frac{a_n F^{n+1} + a_{n+1} F^{n} + \cdots + a_1 F^2 + a_2 F}{a_1} \right) X, Y = 0 \)

Proof. a) => b)

\[
mN(X,Y) = 0 \]

\[
=> m\{ [FX,FY] + F^2 [X,Y] - F[FX,Y] - F[X,FY] \} = 0 \]

\[
=> m[F,F] = 0 \]

\[\text{[since } mF = Fm = 0 \text{]} \]

c) => a)

\[
mN\left( \frac{a_n F^{n+1} + a_{n+1} F^{n} + \cdots + a_1 F^2 + a_2 F}{a_1} \right) X, Y = 0 \]

By (1)

\[
\Rightarrow mX, Y = 0 \]

\[
=> mN[X,Y] = 0 \]

\[
=> m[X,Y] = 0 \]

\[
=> c) => a) \]

Theorem 8. If \( F^n \) acts on \( D_1 \) as an almost complex structure. Then

\[
m\left( \frac{a_n F^{n+1} + a_{n+1} F^{n} + \cdots + a_1 F^2 + a_2 F}{a_1} \right) X, Y = m[-X,FY] = 0 \] (20)

Proof.

\[
m\left( \frac{a_n F^{n+1} + a_{n+1} F^{n} + \cdots + a_1 F^2 + a_2 F}{a_1} \right) X, Y = m[-X,FY] = m[-X,FY] \] 

\[\Rightarrow m[-X,FY] = 0 \text{ [By (1)]} \]

\[
=> mFX, FY = 0 \]

\[\Rightarrow e) => b) \]

Theorem 9. For \( X, Y \in x(D_1) \) we have

\[l\{ (X,Y) + [FX,FY] \} = [X,Y] + [FX,Y] \]

Proof.

\[l\{ (X,Y) + [FX,FY] \} = l\{X,Y - FX,Y + FX,Y - YFX\} \]

\[\text{[By (5)]} \]

\[
= X, Y - FX,Y + FX,Y - YFX \]

\[\text{[By (13)]} \]

\[
= [X,Y] + [FX,Y] \]

Theorem 10: The integrable \( F(a_n, a_{n-1}, \ldots, a_1) \) structure satisfying (1) on \( M \) defines a CR-structure \( H \) on it. Such that \( RH = D_1 \).

Proof. From theorem 4 we have,

\[ [P, Q] = [X,Y] - [FX,FY] - \sqrt{-1} [X,Y] - \sqrt{-1} [FX,Y] \]
\[\{P, Q\} = [X, Y] - [FX, FY] - \sqrt{-1}([X, FY] + [FX, Y])\]  
(By theorem (9))

\[= [X, Y] - [FX, FY] - \sqrt{-1}([X, FY] + [FX, Y]) = [P, Q]\]  
(By theorem (4))

Since \(\{P, Q\} = [P, Q] \Rightarrow [P, Q] \in \xi(D)\). Then, \(F(a_n, a_{n-1}, \ldots, a_i)\) structure satisfying (1) on \(M\) defines a CR-structure.

V. MORPHISM OF VECTOR BUNDLES

Let \(\bar{K}\) be the complementary distribution of \(\text{Re}(H)\) to \(TM\). We define a morphism of vector bundles \(F: TM \rightarrow TM\) given by

\[F(X) = 0 \forall X \in \xi(\bar{K})\]  
(such that-

We have

\[F(X) = \frac{1}{2} \sqrt{-1}(P - \bar{P})\]

where \(P = X + \sqrt{-1}Y \in \xi(HP)\) and \(\bar{P}\) is the complex of \(P\).

**Corollary 1.** If \(P = X + iY\) and \(\bar{P} = X - iY\) belong to \(H_p\) and \(F(X) = \frac{1}{2}\sqrt{-1}(P - \bar{P})\), \(F(Y) = \frac{1}{2}\sqrt{-1}(P + \bar{P})\) and \(F(-Y) = \frac{1}{2}\sqrt{-1}(P + \bar{P})\) then \(F(X) = \frac{1}{2}\sqrt{-1}(P - \bar{P}) = -Y\), \(F(2X) = X\), and \(F(-Y) = -X\).

**Proof.** \(P = X + \sqrt{-1}Y\) and \(\bar{P} = X - \sqrt{-1}Y \Rightarrow (P + \bar{P}) = \frac{(P - \bar{P})}{\sqrt{-1}} = Y\). Since \(P + \bar{P} = 2X\) and \(P - \bar{P} = 2\sqrt{-1}Y\), \(F(X) = F\left(\frac{(P + \bar{P})}{\sqrt{-1}}\right) = \frac{1}{2}\sqrt{-1}(P - \bar{P}) = -Y\) from the definition of morphism

\[F(-Y) = F\left(\frac{P - \bar{P}}{\sqrt{-1}}\right) = -X\]

**Theorem 11.** If \(M\) has a CR-structure \(H\), then we have \(a_n F^n + a_{n-1} F^{n-1} \ldots a_2 F^2 + a_1 F^1 = 0\) and consequently \(F(a_n, a_{n-1}, \ldots, a_2, a_1)\) structure satisfying (1) is defined on \(M\) such that the distribution \(D_1\) and \(D_2\) coincide with \(\text{Re}(H)\) and \(\bar{K}\) respectively.

**Proof.** Suppose \(M\) has a CR-structure. Then in view of definition of CR manifold & corollary 1 we have-

\[F(X) = -Y;\]

operating above equation by \(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} \ldots a_2 F^2}{a_1}\) on both sides we get

\[\left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} \ldots a_2 F^2}{a_1}\right) F(X) = \left(\frac{a_n F^{n-1} + a_{n-1} F^{n-2} \ldots a_2 F^2}{a_1}\right) (-Y)\]

on making use of Corollary 1 the right hand side of the above equation becomes

\[\frac{a_n F^{n-1} + a_{n-1} F^{n-2} \ldots a_2 F^2}{a_1} F(X) = \frac{a_n F^{n-1} + a_{n-1} F^{n-2} \ldots a_2 F^2}{a_1} (-Y)\]

which can be written as–

\[-\frac{a_n F^{n-1} + a_{n-1} F^{n-2} \ldots a_2 F^2}{a_1} F(X) = -\frac{a_n F^{n-1} + a_{n-1} F^{n-2} \ldots a_2 F^2}{a_1} (-X)\]

On simplifying the above equation we get

\[a_n F^n + a_{n-1} F^{n-1} \ldots a_2 F^2 + a_1 F^1 = 0\]

**REFERENCES**

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