Rayleigh-Bénard-Taylor Convection of Newtonian Nanoliquid

P. G. Siddheshwar, T. N. Sakshath

Abstract—In the paper we make linear and non-linear stability analyses of Rayleigh-Bénard convection of a Newtonian nanoliquid in a rotating medium called Rayleigh-Bénard-Taylor convection. Rigid-rigid isothermal boundaries are considered for investigation. Khanfer-Vafai-Lightstone single phase model is used for studying instabilities in nanoliquids. Various thermophysical properties of nanoliquid are obtained using phenomenological laws and mixture theory. The primary boundary value problem is solved for the Rayleigh number using an analytical method by considering trigonometric eigen functions. We observe that the critical nanoliquid Rayleigh number is less than that of the base liquid. Thus the onset of convection is advanced due to the addition of nanoparticles. So, increase in volume fraction leads to advanced onset and thereby increase in heat transport. The amplitudes of convective modes required for estimating the heat transport are determined analytically. The tri-modal standard Lorenz model is derived for the steady state assumption small scale convective motions. The effect of rotation on the onset of convection and on heat transport is investigated and depicted graphically. It is observed that the onset of convection is delayed due to rotation and hence leads to decrease in heat transport. Hence, rotation has a stabilizing effect on the system. This is due to the fact that the energy of the system is used to create the component V. We observe that the amount of heat transport is less in the case of rigid-rigid isothermal boundaries compared to free-free isothermal boundaries.

Keywords—Nanoliquid, rigid-rigid, rotation, single-phase.

NOMENCLATURE

Latin symbols

- $C_p$: specific heat at constant pressure
- $g'$: acceleration due to gravity (0, 0, $-g$)
- $h$: distance between the plates
- $k$: thermal conductivity
- $p$: pressure
- $q'$: velocity vector ($u$, $v$, $w$)
- $T$: dimensional temperature
- $T_0$: temperature of the upper plate (reference temperature)
- $w$: dimensional vertical velocity component
- $U$: non-dimensional horizontal velocity component
- $W$: non-dimensional vertical velocity component
- $x$: dimensional horizontal coordinate
- $z$: dimensional vertical coordinate
- $X$: non-dimensional horizontal coordinate
- $Z$: non-dimensional vertical coordinate

Greek symbols

- $\alpha$: thermal diffusivity
- $\beta$: thermal expansion coefficient
- $\chi$: volume fraction
- $\Delta T$: temperature difference
- $\mu$: dynamic viscosity
- $\nabla^2$: Laplacian operator ($\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$)
- $\psi$: stream function
- $\rho$: density
- $\Theta$: non-dimensional temperature

Subscripts

- $b$: basic state
- $l$: base liquid
- $nl$: nanoliquid
- $np$: nanoparticle
- $c$: critical

I. INTRODUCTION

The effect of rotation is shown to have a significant impact on the flow in porous media. The effect of Coriolis force on the onset of convection and extent to which it delays the onset of convection was examined by Chandrasekhar [8]. Experimental study that includes the stability of Rayleigh-Bénard convection over a wide range of Taylor numbers was conducted by Rossby [17]. Liu and Ecke [13] presented the experimental studies of turbulent thermal convection in water confined in a cell with a square cross section with and without rotation. Some of the recent developments in bifurcation theory and their relevance to the study of rotating convection was summarized by Knobloch [12]. Thermal instabilities of a fluid contained in rotating system are investigated by Busse [6]. Agarwal et al. [1] studied the thermal instability in a rotating anisotropic porous layer saturated by a nanofluid while the thermal instability in a rotating porous layer considering the effect of Brownian motion and thermophoresis was discussed by Bhadouria and Agarwal [3]. Galdi and Straughan [10] applied the nonlinear energy stability theory to study the stabilizing effect of rotation. Vadasz [20] carried out an analytical investigation of the Coriolis effect on three dimensional gravity-driven convection in a rotating porous layer using linear and weakly non linear stability theories. Beaume et al. [2] computed the non-linear solutions of the equations describing two-dimensional convection in a rotating horizontal layer with constant angular velocity. The stability of a rotating doubly diffusive fluid was studied by Pearlstein [15]. The influence of various parameters on convection in the presence of rotation, for both high and low rotation rates was discussed...
by Vanishree and Siddheshwar [21]. The effect of modulation of the rotation speed on the Rayleigh-Bénard instability was investigated by Bhattacharjee [4]. Cox and Matthews [9] described new instabilities arising in three related convection problems namely rotating convection, magnetoconvection and rotating magnetoconvection. Riahi [16] discussed the effect of gravity on Rayleigh-Bénard-Taylor convection are:

a) Phenomenological laws

\[ \frac{k_{nl}}{\mu_1} = \frac{1}{(1-\chi)^{2.5}} \] (Brinkman model [5]).

\[ \frac{k_{nl}}{\mu_1} = \left( \frac{k_{np}}{\mu_1} + 2 \right) - 2\chi \left( 1 - \frac{k_{np}}{\mu_1} \right) \] (Hamilton-Crosser model [11]).

b) Mixture theory

\[ \alpha_{nl} = \frac{k_{nl}}{(\rho C_p)_{nl}}, \quad \rho_{nl} = (1 - \chi + \chi \frac{\rho_{np}}{\rho_1}, \]

\[ (\rho C_p)_{nl} = (1 - \chi + \chi \frac{(\rho C_p)_{np}}{(\rho C_p)_l}), \quad (\rho \beta)_{nl} = (1 - \chi + \chi \frac{(\rho \beta)_{np}}{(\rho \beta)_l}) \] .

1) Basic State Solution: We assume boundary conditions on \( \vec{q} \) and T to be:

\[ \vec{q} = 0, \quad T = T_0 + \Delta T \text{ at } z = \frac{h}{2}, \]
\[ \vec{q} = 0, \quad T = T_0 \text{ at } z = \frac{h}{2}. \]

Taking the velocity, temperature, density and pressure in the quiescent basic state as follows:

\[ \vec{q} = \vec{q}_b = (0, 0), \quad p(z) = p_b(z), \quad \rho(z) = \rho_b(z), \quad T(z) = T_b(z) \]

we obtain the quiescent state solution for the temperature in the form:

\[ T_b(z) = T_0 + \Delta T \left( 1 - \frac{z}{h} \right). \]

We now superimpose perturbations on the quiescent basic state and so we write:

\[ \vec{q} = \vec{q}_b + \vec{q}', \quad p = p_b + p', \quad \rho = p_b + \rho', \quad T = T_b + T' \]

where the primes indicate a perturbed quantity. Now eliminating the pressure \( p \) between the x- and y-components of (2), introducing the stream function \( \psi(x,z) \) in the form

\[ u = \frac{\partial \psi}{\partial z} \quad \text{and} \quad w = -\frac{\partial \psi}{\partial x}, \]

and incorporating the quiescent state solution and non dimensionalizing the resulting equations as well as (3) using the following definition

\[ (X,Z) = \left( \frac{x}{h}, \frac{z}{h} \right), \quad \Psi = \psi \alpha_i, \quad \Theta = \frac{T}{\Delta T}, \quad V = \frac{v h}{\alpha_i}, \]

\[ U = \frac{u h}{\alpha_i}, \quad W = \frac{w h}{\alpha_i}, \]

we obtain the dimensionless form of governing equations as:

\[ \alpha_1 \nabla^4 \Psi - a_1^2 R a_{nl} \frac{\partial \Theta}{\partial X} + a_1 \sqrt{T} \frac{\partial V}{\partial Z} + \frac{1}{Pr_{nl}} \frac{\partial (\Psi, \nabla^2 \Psi)}{\partial (X, Z)} = 0, \]

\[ -\frac{\partial \Psi}{\partial X} + a_1 \nabla^2 \Theta + \frac{\partial (\Psi, \Theta)}{\partial (X, Z)} = 0, \]

A. Mathematical Formulation

The schematic of the flow configuration is as shown in Fig. 1. The coordinate system is taken at the lower boundary with the z-axis taken vertically upwards and the x-axis along the plates. The system is rotated about the z-axis with uniform angular velocity \( \Omega \).

The governing equations describing Rayleigh-BénardTaylor convection are:

\[ \nabla \cdot \vec{q} = 0, \]

\[ \rho_{nl}(\vec{q} \cdot \nabla) \vec{q} = -\nabla p + \rho_{nl} \nabla^2 \vec{q} + \rho_{nl} \vec{q} + 2\rho_{nl}(\vec{q} \times \Omega). \]

\[ \alpha_{nl} \nabla^2 T = (\vec{q} \cdot \nabla)T, \]

\[ \rho_{nl}(T) = \rho_{nl}(T_0) - (\rho \beta)_{nl}(T - T_0), \]

where the nanoliquid properties are obtained from either phenomenological laws or mixture theory as given below:

The schematic of the flow configuration is as shown in Fig. 1. The coordinate system is taken at the lower boundary with the z-axis taken vertically upwards and the x-axis along the plates. The system is rotated about the z-axis with uniform angular velocity \( \Omega \).

\[ \vec{q} \]

Cold, \( T_0 \)

Newtonian liquid with nanoparticles

Hot, \( T_0 + \Delta T \)

\[ z = \frac{h}{2} \]

\[ x \]

Fig. 1 Schematic representation of Rayleigh-Bénard-Taylor convection of Newtonian nanoliquid
\[ \nabla^2 V - \sqrt{T_0} \frac{\partial \Psi}{\partial Z} + \frac{1}{Pr_{nl}} \frac{\partial (\Psi, V)}{\partial X} = 0, \quad (13) \]

where \( V \) is the velocity in the \( y \) direction which vary along \( x \) and \( z \) directions,

\[
a_1 = \frac{\alpha}{\alpha_1} \text{ (thermal diffusivity ratio)},
\]

\[
Ra_{nl} = \left( \frac{\rho \beta a h^3 g \Delta T}{\mu_{nl} \alpha_{nl}} \right) \text{ (nanoliquid Rayleigh number)},
\]

\[
Ta = \left( \frac{2 \rho_{nl} \Omega h^2}{\phi_{nl}} \right)^2 \text{ (modified Taylor number)},
\]

\[
Pr_{nl} = \frac{\mu_{nl}}{\rho_{nl} \alpha_{nl}} \text{ (nanoliquid Prandtl number)}.
\]

In the next section, we make a linear stability analysis and study the onset of convection.

### B. Linear Stability Analysis

The boundary conditions suitable for rigid-rigid isothermal boundaries are:

\[
\Psi = \frac{\partial \Psi}{\partial Z} = \Theta = V = 0 \text{ at } Z = \pm \frac{1}{2}. \quad (14)
\]

The normal mode solution for solving eigen boundary value problem is:

\[
\Psi = A \sin(\nu X) (C_f)_{nc}(Z), \quad \Theta = B \cos(\nu X) \sin[\pi(Z + \frac{1}{2})], \quad V = C \sin(\nu X) \z \sin[\pi(Z + \frac{1}{2})], \quad \left\{ \right. 
\]

where \( A, B \) and \( C \) are the amplitudes, \( \nu \) is the wave number and \((C_f)_{nc}(Z)\) is the Chandrasekhar function (even solution) \([7]\) and \( \mu_1 = 4.73004074 \) is the eigen value satisfying the following equation,

\[
tanh\left( \frac{\mu_1}{2} \right) + \tan\left( \frac{\mu_1}{2} \right) = 0. \quad (16)
\]

Substituting (15) in the dimensionless form of the governing equations (11)-(13) and following the standard orthogonalization procedure, we obtain the expression of the critical value of nanoliquid Rayleigh number for stationary onset as:

\[
Ra_{nl c} = \frac{\delta^2 (F_1 (\nu_c^3 + \mu_1^2) + 2 F_2 \nu_c^2 \mu_1^2)}{2 F_C^2 \nu_c^2 \delta^2 Ta} + \frac{F_2^2 \mu_1^2}{12 \pi^4 F_C^2} \left( -6 \nu_c^2 \pi^2 (\mu_1^2 + 6) + \pi^4 \right) \quad (17)
\]

where

\[
\delta^2 = \nu_c^2 + \pi^2,
\]

\[
F_1 = \frac{1}{1 + \cos(\mu_1)} - \frac{\tan\left( \frac{\mu_1}{2} \right)}{\mu_1} \quad \text{ and } \quad F_2 = \frac{1}{1 + \cos(\mu_1)} - \frac{1}{1 + \cosh(\mu_1)}.
\]

The truncated representation for making a weakly non-linear analysis for rigid-rigid, isothermal boundaries is:

\[
\Psi = A \sin(\nu_c X) (C_f)_{nc}(Z), \quad \Theta = B \cos(\nu_c X) \sin[\pi(Z + \frac{1}{2})] + C \sin[2\pi(Z + \frac{1}{2})], \quad V = D \sin(\nu_c X) \z \sin[\pi(Z + \frac{1}{2})]. \quad (22)
\]

Substituting (22) into (11)-(13) and using the orthogonality condition with the eigen functions on the resulting equations, we get the following algebraic equations:

\[
2a_{1} \nu_c \left[ F_1 \left( \nu_c^3 + \frac{\mu_1^2}{\nu_c} \right) + 2 F_2 \nu_c \mu_1^2 \right] A - 2a_{1}^2 F_3 \nu_c R a_{nl} B + 2\pi a_1 F_4 \sqrt{T a} D = 0, \quad (23)
\]

\[
A \left( \frac{F_5 C + 2 F_3 }{a_{1} \nu_c^2} \right) - B = 0, \quad (24)
\]

\[
AB + \frac{8 a_{1} \pi^2}{F_5 \nu_c} C = 0, \quad (25)
\]

\[
2\pi F_4 \sqrt{T a} A - \frac{\left[ -6 \nu_c^2 \pi^2 (6 + \delta^2) \right]}{12 \pi^2} D = 0. \quad (26)
\]

where \( F_1, F_2, F_3, F_4 \) are given by (18)-(21) and

\[
F_5 = \frac{16 \pi^2 \mu_1^2 \left( \mu_1^2 + 39 \pi^4 \right)}{\mu_1^8 - 82 \pi^4 \mu_1^4 + 81 \pi^8}. \quad (27)
\]
Solving (23)-(26), we get

\[
A^2 = \frac{8\pi^2 \delta_2 a_1^2 r}{\nu F_3} \left[ 1 - \frac{1}{r} \right],
\]

(28)

\[
B = \frac{2\nu F_3}{a_1 \delta_1^2 r},
\]

(29)

\[
C = -\frac{F_3 \nu^2}{4\pi^2 a_1^2 \delta_1^2 r} A^2 - \frac{2F_3}{F_5} \left[ 1 - \frac{1}{r} \right],
\]

(30)

\[
D = \frac{24\pi^3 F_3 \sqrt{T_a}}{-6\nu_c^2 + \pi^2 (6 + \delta_c^2) A},
\]

(31)

where

\[
r = \frac{Ra_{nl}}{Ra_{nlc}}
\]

(32)

is the scaled Rayleigh number.

We next study the heat transport in terms of Nusselt number.

D. Nusselt Number

The amount of heat transport by Rayleigh-Bénard-Taylor convection for rigid-rigid, isothermal boundaries can be quantified in terms of the Nusselt number, \( Nu_{nl} \), as follows:

\[
Nu_{nl} = \frac{\text{Heat transport by (conduction + convection)}}{\text{Heat transport by conduction}}.
\]

Using Fourier first law and further simplifying, we get:

\[
Nu_{nl} = 1 + \frac{k_{nl}}{k_l} \left[ -\int_0^{\frac{\pi}{2}} \frac{\partial \Theta}{\partial Z} dX \right] - \frac{k_{nl}}{k_l} \left[ \int_0^{\frac{\pi}{2}} \frac{\partial \Theta}{\partial Z} dX \right]_{Z = -\frac{1}{2}}
\]

(33)

where \( k_{nl} \) and \( k_l \) are the thermal conductivities of the nanoliquid and base liquid respectively.

Substituting dimensionless form of (9) and (22) in (33) and completing the integration, we get

\[
Nu_{nl} = 1 - 2\pi \frac{k_{nl}}{k_l} C.
\]

(34)

Using (30), (33) takes the form

\[
Nu_{nl} = 1 + 2 \frac{2\pi F_3}{F_5} \frac{k_{nl}}{k_l} \left[ 1 - \frac{1}{r} \right],
\]

where \( F_3, F_5 \) are given by (20) and (27) and \( r \) is given by (32).

II. Conclusion

From Tables I and II it is found that the thermophysical properties of the base liquid, nanoliquid and nanoparticles vary as shown below:

a) \( k_l < k_{nl} << k_{np} \),

b) \( (C_p)_{np} > (C_p)_{l} > (C_p)_{nl} \),

c) \( \rho_{np} > \rho_{nl} > \rho_{l} \),

d) \( \beta_l > \beta_{nl} > \beta_{np} \),

e) \( \alpha_{np} >> \alpha_{nl} > \alpha_{l} \).

To study the implications of linear stability results, we may write \( Ra_{nl} = FRa_l \),

where

\[
F = \frac{(\rho_3^2)_{nl}}{(\rho_3^2)_{l}} \frac{\mu_l}{\mu_{nl}} \frac{\alpha_l}{\alpha_{nl}}\text{ and } Ra_l = \frac{\rho_l^2 \gamma g h^3 \Delta T}{\mu_l \alpha_l}.
\]

On further computation it is seen that the factor, \( F \), multiplying \( Ra_l \) decreases with increase in \( \chi \). This leads to the conclusion that the critical value of nanoliquid Rayleigh number is less than that of the base liquid without nanoparticles.

Rotation delays the onset of convection and thereby decreases heat transport. This result is shown in Fig. 2. This is because the energy of the system is used to create the component V.

The amount of heat transport increases with increase in \( \chi \) and this is depicted in Fig. 3. Increase in the value of \( \chi \) implies increase in volume fraction of nanoparticles.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Ethylene Glycol</th>
<th>Copper nanoparticles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>( \rho_l = 1114.4 )</td>
<td>( \rho_{np} = 8933 )</td>
</tr>
<tr>
<td>Specific heat</td>
<td>( (C_p)_l = 2415 )</td>
<td>( (C_p)_{np} = 385 )</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>( k_l = 0.252 )</td>
<td>( k_{np} = 401 )</td>
</tr>
<tr>
<td>Thermal expansion coefficient</td>
<td>( \beta_l = 65 )</td>
<td>( \beta_{np} = 1.67 )</td>
</tr>
<tr>
<td>Dynamic coefficient of viscosity</td>
<td>( \mu_l = 0.0157 )</td>
<td>-</td>
</tr>
<tr>
<td>Thermal diffusivity</td>
<td>( \alpha_l = 0.93636 )</td>
<td>( \alpha_{np} = 1165.9 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Ethylene glycol-Copper nanoparticles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>1896.26</td>
</tr>
<tr>
<td>Specific heat</td>
<td>1458.70</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>0.335824</td>
</tr>
<tr>
<td>Thermal expansion coefficient</td>
<td>35.1662</td>
</tr>
<tr>
<td>Dynamic coefficient of viscosity</td>
<td>0.020431</td>
</tr>
<tr>
<td>Thermal diffusivity</td>
<td>1.21408</td>
</tr>
<tr>
<td>[J/m s K]</td>
<td>2.76607</td>
</tr>
<tr>
<td>[kg/m s K]</td>
<td>0.666842</td>
</tr>
</tbody>
</table>

The thermophysical properties of Ethylene Glycol and Copper Nanoparticles at 300K for Volume Fraction, \( \chi = 0.1 \) [18]

The thermophysical properties of Ethylene Glycol-Copper Nanoliquid at 300K for Volume Fraction, \( \chi = 0.1 \) [18]
ACKNOWLEDGMENT

One of the authors (TNS) would like to thank the Department of Backward Classes Welfare, Government of Karnataka for providing fellowship to carry out his research work. The authors would like to thank the Bangalore University for their support.

REFERENCES


P. G. Siddheshwar is a Professor of Mathematics at Bangalore University. He has more than 100 research papers at his credit on the topic of linear and non-linear stability of natural convection in Newtonian and non-Newtonian clear fluids and nanoliquids in saturated porous medium with local thermal equilibrium and local thermal non-equilibrium.

T. N. Sakshath is a Doctoral student of Mathematics at Bangalore University. He is in the third year of his research period. His research topic is Rayleigh Bénard convection of nanoliquids in porous media.