Abstract—The aim of this paper is to introduce the concepts of \((\epsilon,\epsilon\lor q)\)-fuzzy subalgebras, \((\epsilon,\epsilon\lor q)\)-fuzzy ideals and \((\epsilon,\epsilon\lor q)\)-fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

Keywords—BCI-algebras with operators, \((\epsilon,\epsilon\lor q)\)-fuzzy subalgebras, \((\epsilon,\epsilon\lor q)\)-fuzzy ideals, \((\epsilon,\epsilon\lor q)\)-fuzzy quotient algebras.

I. INTRODUCTION

The fuzzy set is a generalization of the classical set and was used afterwards by several authors such as Imai [1], Iseki [2] and Xi [3], in various branches of mathematics. Particularly, in the area of fuzzy topology, after the introduction of fuzzy sets by Zadeh [15], much research has been carried out: the concept of fuzzy subalgebras and fuzzy ideals of BCK-algebras, and their some properties.


In this paper, we introduce the concepts of \((\epsilon,\epsilon\lor q)\)-fuzzy subalgebras, \((\epsilon,\epsilon\lor q)\)-fuzzy ideals and \((\epsilon,\epsilon\lor q)\)-fuzzy quotient algebras of BCI-algebras with operators. Moreover, the basic properties were discussed and several results have been obtained.

II. PRELIMINARIES

Some definitions and propositions were recalled which may be needed.

An algebra \(\{x,\ast,0\}\) of type \((2,0)\) is called a BCI-algebra, if for all \(x, y, z \in X\), it satisfies:

\[ (1) ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0; \]
\[ (2) \ast (x \ast y) \ast y = 0; \]
\[ (3) x \ast x = 0; \]
\[ (4) x \ast y = 0 \text{ and } y \ast x = 0 \text{ imply } x = y. \]

We can define \(x \ast y = 0\) if and only if \(x \leq y\), and the above conditions can be written as:

1. \((x \ast y) \ast (x \ast z) \leq z \ast y;\)
2. \(x \ast (x \ast y) \leq y;\)
3. \(x \leq x;\)
4. \(x \leq y\) and \(y \leq x\) imply \(x = y.\)

A BCI-algebra is called a BCK-algebra if it satisfies \(0 \ast x = 0\).

Definition 1. [5] \(\{X,\ast,0\}\) is a BCI-algebra, a fuzzy subset \(A\) of \(X\) is called a fuzzy ideal of \(X\) if it satisfies:

\[ (1) A(0) \geq A(x), \forall x \in X, \]
\[ (2) A(x) \geq A(x \ast y) \land A(y), \forall x, y \in X. \]

Definition 2. [4] \(\{X,\ast,0\}\) is a BCI-algebra, a fuzzy subset \(A\) of \(X\) is called a fuzzy subalgebra of \(X\) if it satisfies:

\[ A(x \ast y) \geq A(x) \ast A(y), \forall x, y \in X. \]

Definition 3. [12] \(\{X,\ast,0\}\) is a BCI-algebra, a fuzzy subset \(A\) of \(X\) of the form

\[ A(y) = \begin{cases} t, & y = x, \\ 0, & x \neq y, \end{cases} \]

is said to be a fuzzy point with support \(x\) and value \(t\), and is
denoted by $x_r$.

**Definition 4.** [12] If $x_r$ is a fuzzy point, it is said to belong to (resp. be quasi-coincident with) a fuzzy subset $A$, written as $x_r \in A$ (resp. $x_r \in q A$) if $A(x) \geq t$ (resp. $A(x) + t > 1$). If $x_r \in A$ or $x_r \in q A$, then we write $x_r \in v q A$. The symbol $\in v q$ (resp. $\in q$) means $\in v q$ (resp. $\in q$) does not hold.

**Definition 5.** [10] $\langle X; \ast, 0 \rangle$ is a BCI-algebra, a fuzzy set $A \subseteq X$ is called an $(\epsilon, \epsilon \in v q)$-fuzzy ideal of $X$ if for all $t, r \in (0, 1]$ and $x_r, y \in X$, it satisfies:

1. $x_r \in A \Rightarrow 0_r \in v q d$,
2. $(x_r \ast y) \in A$ and $y \in A \Rightarrow x_r \in v q A$.

**Definition 6.** [10] A fuzzy set $A$ is an $(\epsilon, \epsilon \in v q)$-fuzzy ideal of $X$ if and only if it satisfies:

1. $A(0) \geq A(x) \vee 0_r, \forall x \in X$,
2. $A(x) \geq A(x \ast y) \ast A(y) \vee 0_r, \forall x, y \in X$.

**Definition 7.** [7] $\langle X; \ast, 0 \rangle$ is a BCI-algebra, $M$ is a non-empty set, if there exists a mapping $(m, x) \rightarrow mx$ from $M \times X$ to $X$ which satisfies

$$m(x \ast y) = (mx) \ast (my), \forall x, y \in X, m \in M,$$

then $M$ is called a left operator of $X$, $X$ is called BCI-algebra with left operator $M$, or $X$ is a BCI-algebra for short.

**Proposition 1.** [6] Let $\langle X; \ast, 0 \rangle$ be a BCI-algebra, if $A$ is an $(\epsilon, \epsilon \in v q)$-fuzzy ideal of it, and $x_r \ast y \leq z$, then

$$A(x) \geq A(y) \ast A(z) \vee 0_r, \forall x, y, z \in X.$$

**Definition 8.** [13] Let $A$ and $B$ be fuzzy sets of set $X$, then the direct product $A \times B$ of $A$ and $B$ is a fuzzy subset of $X \times X$, define $A \times B$ by

$$A \times B(x, y) = A(x) \ast B(y), \forall x, y \in X.$$

**Definition 9.** [7] Let $\langle X; \ast, 0 \rangle$ and $\langle \overline{X}; \overline{\ast}, 0 \rangle$ be two $M$-BCI-algebras, if for all $x \in X, m \in M$, $f(mx) = mf(x)$, and $f$ is a homomorphism from $\langle X; \ast, 0 \rangle$ to $\langle \overline{X}; \overline{\ast}, 0 \rangle$, then $f$ is called a homomorphism with operators.

**Definition 10.** [13] $\langle X; \ast, 0 \rangle$ is an $M$-BCI-algebra, let $B$ be a fuzzy set of $X$, and $A$ be a fuzzy relation of $B$, if it satisfies:

$$A_B(x, y) = B(x) \ast B(y), \forall x, y \in X,$$

then $A$ is called a strong fuzzy relation of $B$.

**Definition 11.** [14] If $\langle X; \ast, 0 \rangle$ is an $M$-BCI-algebra, $A$ is a non-empty subset of $X$, and $m \in A$ for all $x \in A$, $m \in M$, then $\langle A, x, 0 \rangle$ is called a $M$-subalgebra of $\langle X; \ast, 0 \rangle$.

In this paper, $X$ always means a $M$-BCI-algebra unless otherwise specified.

**III. $(\epsilon, \epsilon \in v q)$-FUZZY SUBALGEBRAS OF BCI-ALGEBRAS WITH OPERATORS**

**Definition 12.** $\langle X; \ast, 0 \rangle$ is a BCI-algebra, a fuzzy set $A$ of $X$ is called a $M - (\epsilon, \epsilon \in v q)$-fuzzy subalgebra of $X$ if for all $t, r \in (0, 1]$ and $x_r, y \in X$, it satisfies:

1. $x_r \in A$ and $y \in A \Rightarrow (x_r \ast y) \in v q A$,
2. $x_r \in A \Rightarrow (mx) \in v q A$.

**Proposition 2.** $\langle X; \ast, 0 \rangle$ is a BCI-algebra, a fuzzy set $A$ of $X$ is an $M - (\epsilon, \epsilon \in v q)$-fuzzy subalgebra of $X$ if and only if it satisfies:

1. $A(x \ast y) \geq A(x) \ast A(y) \vee 0_r, \forall x, y \in X$,
2. $A(mx) \geq A(x) \vee 0_r, \forall x \in X$.

**Proof.** Suppose that $A$ is an $M - (\epsilon, \epsilon \in v q)$-fuzzy subalgebra of $X$. (1) Let $x, y \in X$, suppose that $A(x) \ast A(y) < 0$, then $A(x \ast y) < A(x) \ast A(y)$, if not, then we have $A(x \ast y) < A(x) \ast A(y)$, $\exists \epsilon \in (0, 0.5)$, it follows that $x_r \in A$ and $y \notin A$, but $(x_r \ast y)_{\epsilon, \epsilon} \in v q A$, which is a contradiction, otherwise $A(x) \ast A(y) < 0$. We have $A(x \ast y) \geq A(x) \ast A(y)$. If $A(x) \ast A(y) \geq 0$, then $x_r \ast y \in A$, which implies that $(x \ast y)_{\epsilon, \epsilon} \in v q A$, therefore $A(x \ast y) \geq 0$, because if $A(x \ast y) < 0$, then $A(x \ast y) + 0.5 < 0 + 0.5 = 1$, which is a contradiction, hence

$$A(x \ast y) \geq A(x) \ast A(y) \vee 0_r, \forall x, y \in X.$$
\[ A(x * y) + t_1 \wedge t_2 \geq \text{2.} \]  
so that \((x * y)_{t_1, t_2} \in v^{-1}Q A.
\]

(2) Let \(x \in X\) and \(t \in [0,1]\) be such that \(x, t \in A\), then we have \(A(x) \geq t\). Suppose that \(A(mx) < t\) and \(A(mx) \geq A(x) \wedge 0.5 \geq A(x) \geq t\), this is a contradiction, hence we know that \(A(x) \geq 0.5\), and we have:

\[ A(mx) + t > 2(A(mx)) \geq 2(A(x) \wedge 0.5) = 1, \]

then \((mx) \in v^{-1}Q A\). Consequently, \(A\) is an \(M - (e, e_v)\) fuzzy subalgebra.

**Example 1.** If \(A\) is an \(M - (e, e_v)\) fuzzy subalgebra of \(X\), then \(X_d\) is an \(M - (e, e_v)\) fuzzy subalgebra of \(X\), define \(X_d\) by:

\[ X_d : X \rightarrow [0,1], X_d(x) = \begin{cases} 1, x \in A, \\ 0, x \notin A. \end{cases} \]

**Proof.** (1) For all \(x, y \in X\), if \(x, y \in A\), then \(x * y \in A\), then we have:

\[ X_d(x * y) = 1 \geq X_d(x) \wedge X_d(y) \wedge 0.5, \]

if there exists at least one which does not belong to \(A\) between \(x\) and \(y\), for example \(x \notin A\), thus:

\[ X_d(x * y) = 0 = X_d(x) \wedge X_d(y) \wedge 0.5. \]

(2) For all \(x \in X, m \in M\), if \(x \in A\), then \(mx \in A\), therefore:

\[ X_d(mx) = 1 \geq X_d(x) \wedge 0.5, \]

if \(x \notin A\), then \(X_d(mx) = 0 \geq X_d(x) \wedge 0.5\), therefore \(X_d\) is an \((e, e_v)\) fuzzy subalgebra of \(X\).

**Proposition 3.** A is an \(M - (e, e_v)\) fuzzy subalgebra of \(X\) if and only if \(A\) is an \(M\) subalgebra of \(X\), where \(A\) is a non-empty set, define \(X_d\) by:

\[ A = \{ x \in X, A(x) \geq t \}, \forall t \in [0,0.5]. \]

**Proof.** Suppose \(A\) is an \(M - (e, e_v)\) fuzzy subalgebra of \(X\), \(A\) is a non-empty set, \(t \in [0,0.5]\), then we have \(A(x * y) \geq A(x) \wedge A(y) \wedge 0.5\). If \(x \in A, y \in A\), then:

\[ A(x) \geq t, A(y) \geq t, \]

then we have \(x * y \in A\). If \(A\) is an \(M - (e, e_v)\) fuzzy subalgebra of \(X\), then \(A(mx) \geq A(x) \wedge 0.5 \geq t, \forall x \in X, m \in M\), then we have \(mx \in A\). Therefore \(A\) is an \(M\) subalgebra of \(X\).

Conversely, suppose \(A\) is an \(M - \) subalgebra of \(X\), then we have \(x * y \in A\). Let \(A(x) = t\), then:

\[ A(x * y) \geq t = A(x) \wedge A(y) \wedge 0.5. \]

If \(A\) is an \(M - \) subalgebra of \(X\), then we have:

\[ A(mx) \geq t = A(x) \wedge A(y) \wedge 0.5, \forall x \in X, m \in M, \]

therefore \(A\) is an \(M - (e, e_v)\) fuzzy subalgebra of \(X\).

**Proposition 4.** Suppose \(X, Y\) are \(M\) BCI-algebras, \(f\) is a mapping from \(X\) to \(Y\), if \(A\) is an \(M - (e, e_v)\) fuzzy subalgebra of \(X\), then \(f^{-1}(A)\) is a \(M - (e, e_v)\) fuzzy subalgebra of \(Y\).

**Proof.** Let \(y \in Y\), suppose \(f\) is an epimorphism, and we have \(y = f(x), \exists x \in X\). If \(A\) is an \((e, e_v)\) fuzzy subalgebra of \(Y\), then we have:

\[ A(x * y) \geq A(x) \wedge A(y) \wedge 0.5, \forall x \in X, \]

for all \(x, y \in X, m \in M\), we have:

(1) \(f^{-1}(A) = A(f(x) * f(y)) \geq A(f(x)) \wedge A(f(y)) \wedge 0.5 = f^{-1}(A) \wedge f^{-1}(A) \geq 0.5;
\]

(2) \(f^{-1}(A) = A(f(mx)) = A(mf(x)) \geq A(f(x)) \wedge 0.5 = f^{-1}(A) \wedge 0.5. \)

Then \(f^{-1}(A)\) is an \(M - (e, e_v)\) fuzzy subalgebra of \(X\).

**IV. \((e, e_v)\) Fuzzy Ideals of BCI-Algebras with Operators**

**Definition 13.** \((X, *, 0)\) is a BCI-algebra, a fuzzy set \(A\) of \(X\) is called an \(M - (e, e_v)\) fuzzy ideal of \(X\) if for all \(t, r \in [0,1]\) and \(x, y \in X\), it satisfies:

1. \(x, t \in A \Rightarrow 0 \in vQ A, \)
2. \((x * y) \in A\) and \(y, r \in A \Rightarrow x, r \in vQ A, \)
3. \(x, t \in A \Rightarrow (mx) \in vQ A. \)

**Proposition 5.** \((X, *, 0)\) is a BCI-algebra, a fuzzy set \(A\) is an \(M - (e, e_v)\) fuzzy ideal of \(X\) if and only if it satisfies:

(1) \(A(0) \geq A(x) \wedge 0.5, \forall x \in X, \)
(2) \(A(x) \geq A(x * y) \wedge A(y) \wedge 0.5, \forall x, y \in X, \)
(3) \(A(mx) \geq A(x) \wedge 0.5, \forall x \in X. \)

**Proof.** Suppose that \(A\) is an \(M - (e, e_v)\) fuzzy ideal of \(X\), then...
(1) Let \( x \in X \) and assume that \( A(x) < 0.5 \). If \( A(0) < A(x) \), then we have \( A(0) < A(x) \), which implies that \( A(0) < A(0) \cdot 0.5 \), and we have \( A(0) < A(x) \), and we have \( A(0) < A(x) \), since \( A(0) < A(x) \), we have \( 0 \in A(0) \), which is a contradiction, then \( A(0) \geq A(x) \). Now if \( A(0) < A(x) \), then \( x_0 \in X \), then we have \( 0 \in A(x) \), and \( A(0) \geq A(x) \), otherwise, \( A(0) < 0.5 \), and \( A(0) < A(x) \), which is a contradiction, consequently,

\[ A(0) \geq A(x) \land 0.5, \forall x \in X. \]

(2) Let \( x, y \in X \) and suppose that \( A(x\land y) \land A(y) < 0.5 \), then \( A(x) \geq A(x\land y) \land A(y) \), if not, then we have \( A(x) \land 0.5 < A(x\land y) \land A(y) \), \( \exists \in (0, 0.5) \), it follows that \( \exists \in (0, 0.5) \), which is a contradiction, hence whenever \( A(x\land y) \land A(y) < 0.5 \), we have \( A(x) \land A(y) < 0.5 \), if \( A(x\land y) \land A(y) \geq 0.5 \), then \( x_0 \in X \) and \( y_0 \in X \), which implies that \( x_0 = y_0 \in A(x) \), therefore \( A(x) \geq 0.5 \), because if \( A(x) < 0.5 \), then \( A(x) < 0.5 \), \( 0.5 + 0.5 = 1 \), which is a contradiction, then

\[ A(x) \geq A(x\land y) \land A(y) \land 0.5, \forall x, y \in X. \]

(3) Let \( x \in X \) and assume that \( A(x) < 0.5 \). If \( A(mx) < A(x) \), then we have \( A(mx) < A(x) \), \( \exists \in (0, 0.5) \), and we have \( x_0 \in A(x) \) and \( \exists \in (0, 0.5) \), since \( A(mx) < A(x) \), we have \( \exists \in (0, 0.5) \), \( \exists \in A(x) \), which is a contradiction, then \( A(mx) < A(x) \). Now if \( A(x) \geq 0.5 \), then \( x_0 \in A(x) \), and \( \exists \in (0, 0.5) \), hence \( A(x) \geq 0.5 \), otherwise \( A(mx) < A(x) \), \( 0.5 < 0.5 + 0.5 = 1 \), which is a contradiction, consequently, \( A(mx) \geq A(x) \) \( \forall X \in X \). Conversely, suppose that \( A \) satisfies (1), (2), (3) of the Proposition 5, then we have

(1) Let \( x \in X \) and \( t \in (0, 1) \) be such that \( x \in A \), then we have \( A(x) > t \), suppose that \( A(0) < t \), if \( A(x) < 0.5 \), then \( A(0) \geq A(x) \), \( A(x) > t \), which is a contradiction, then we know that \( A(0) \geq A(x) \), and we have \( A(0) > 0 \), \( A(0) > 0 \), thus \( 0 \in A(0) \), and \( A(0) > 0 \).

(2) Let \( x, y \in X \) and \( t_1, t_2 \in (0, 1) \) be such that \( (x\land y) \in A \), and \( y \in A \), then \( A(x\land y) \geq t_1 \), and \( A(y) \geq t_2 \), suppose that \( A(x) \land A(y) \geq 0.5 \), then

\[ A(x) \geq A(x\land y) \land A(y) \land 0.5 = A(x\land y) \land A(y) \geq t_1 \land t_2, \]

This is a contradiction, so we have \( A(x\land y) \land A(y) \geq 0.5 \), it follows that

\[ A(x) + t_1 \land t_2 > 2A(x) \geq 2A(x) + A(y) \land A(y) \land 0.5 = 1, \]

so that \( x_1 \land t_2 \in \emptyset \).

(3) Let \( x \in X \) and \( t \in (0, 1) \) be such that \( x \in A \), then \( A(x) \geq t \), suppose that \( A(mx) < A(x) \), if \( A(x) < 0.5 \), then \( A(mx) \geq A(x) \), which is a contradiction, then we know that \( A(x) \geq 0.5 \), and we have \( A(mx) > A(x) \), \( 0.5 > 0.5 = 1 \), which is a contradiction, consequently, \( A \) is an \( M(e, e \in [0, 1]) \)-fuzzy ideal.

Example 2. If \( A \) is an \( M(e, e \in [0, 1]) \)-fuzzy ideal of \( X \), then \( X_A \) is an \( M(e, e \in [0, 1]) \)-fuzzy ideal of \( X \), define \( X_A \) by

\[ X_A : X \rightarrow (0, 1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \]

Proof. (1) For all \( x, y \in X \), if \( x, y \in A \), then \( x \land y \in A \), thus

\[ X_A(x) \geq 1 \geq X_A(x) \land 0.5, \]

\[ X_A(x) \geq 1 \geq X_A(x \land y) \land X_A(y) \land 0.5, \]

if there exists at least one between \( x \) and \( y \) which does not belong to \( A \), for example \( x \notin A \), thus

\[ X_A(x) \geq 1 \geq X_A(x) \land 0.5, \]

\[ X_A(x) \geq 1 \geq X_A(x \land y) \land X_A(y) \land 0.5 = 0, \]

therefore \( X_A \) is a \( (e, e \in [0, 1]) \)-fuzzy ideal of \( X \).

(2) For all \( x \in X, m \in M \), if \( x \in A \), then \( mx \in A \), therefore \( X_A(mx) = 1 \geq X_A(x) \land 0.5 \). If \( x \notin A \), then \( X_A(mx) < 0 \), \( X_A(x) \land 0.5 < 0 \), therefore \( X_A \) is an \( M(e, e \in [0, 1]) \)-fuzzy ideal of \( X \).

Proposition 6. \( A \) is an \( M(e, e \in [0, 1]) \)-fuzzy ideal of \( X \) if and only if \( A \) is an \( M \)-ideal of \( X \), where \( A \) is non-empty set, define \( A \) by

\[ A = \{ x \in X, A(x) \geq t \}, \forall t \in [0, 0.5]. \]

Proof. Suppose \( A \) is an \( M(e, e \in [0, 1]) \)-fuzzy ideal of \( X \), \( A \) is non-empty set, \( t \in [0, 0.5] \), then we have \( A(0) \geq 0, A(x) \geq 0 \land 0.5 \), then we have \( 0 \in A \). If \( x \land y \in A \), then \( A(x) \land A(y) \geq 0 \), \( A(x) \land A(y) \geq 0 \), \( A(x) \land A(y) \geq 0 \), \( A(x) \land A(y) \geq 0 \), therefore \( A \) is an \( M(e, e \in [0, 1]) \)-fuzzy ideal of \( X \), hence \( A \) is an \( M \)-ideal of \( X \). Conversely, suppose \( A \) is
an \( M \)-ideal of \( X \), then we have \( 0 \in A, A(0) = t \). Let \( A(x) = t \), thus \( x \in A \), we have \( A(0) = t = A(x) \), suppose there is no \( A(x) \geq A(x^+y) \land A(y) < 0.5 \), then there exist \( x_0, y_0 \in X \), we have \( A(x_0) = A(x_0^+y_0) \land A(y_0) \geq 0.5 \). Let \( t_0 = A(x_0) = A(y_0) < 0.5 \), then \( A(x_0^+y_0) \land A(y_0) \geq 0.5 \), if \( x_0 \cdot y_0 \in A \), then \( A(x_0) \geq t_0 \), which is inconsistent with \( A(x_0) = A(x_0^+y_0) \land A(y_0) \geq 0.5 \). If \( A \) is an \( M \)-ideal of \( X \), then we have \( A(0) = t = A(x) \land A(y) \geq 0.5, \forall x, y \in M \), therefore \( A \) is an \( M-(e, e \in v_q) \)-fuzzy ideal of \( X \).

**Proposition 7.** Suppose \( X, Y \) are \( M \)-BCI-algebras, \( f \) is a mapping from \( X \) to \( Y \), \( A \) is an \( M-(e, e \in v_q) \)-fuzzy ideal of \( Y \), then \( f^{-1}(A) \) is an \( M-(e, e \in v_q) \)-fuzzy ideal of \( X \).

**Proof.** Let \( y \in Y \), suppose \( f \) is an epimorphism, then we have \( y = f(x), \forall x \in X \). If \( A \) is an \( M-(e, e \in v_q) \)-fuzzy ideal of \( Y \), then we have

\[
\begin{align*}
A(0) &\geq A(x) \land A(y) \\
A(x) &\geq A(x \land y) \land A(y) \geq 0.5 \\
A(mx) &\geq A(x) \geq 0.5.
\end{align*}
\]

For all \( x, y \in X, m \in M \), we have

\[
\begin{align*}
(1) \quad f^{-1}(A) = A(f(0)) &\geq A(f(x)) \land A(f(y)) \geq 0.5 \\
(2) \quad f^{-1}(A(x)) &\geq A(f(x) \land f(y)) \geq f^{-1}(A)(x) \land A(y) \geq 0.5 \\
&= A(f(x) \land y) \land A(f(y)) \geq f^{-1}(A)(x \land y) \land f^{-1}(A)(y) \geq 0.5 \\
(3) \quad f^{-1}(A(mx)) &\geq A(mx) \geq A(f(mx)) = A(f(y)) \geq f^{-1}(A)(x) \geq 0.5.
\end{align*}
\]

Therefore \( f^{-1}(A) \) is an \( M-(e, e \in v_q) \)-fuzzy ideal of \( X \).

**V. \((e, e \in v_q)\)-Fuzzy Quotient BCI-Algebras with Operators**

**Definition 14.** Let \( A \) be an \( M-(e, e \in v_q) \)-fuzzy ideal of \( X \), for all \( a \in X \), fuzzy set \( A_a \) on \( X \) defined as: \( A_a : X \rightarrow [0, 1] \)

\[
A_a(x) = A(ax) \land A(x \land a) \geq 0.5, \forall x \in X.
\]

Denote \( X/A = \{ A_a : a \in X \} \).

**Proposition 8.** Let \( A_a, A_b \in X/A \), then \( A_a = A_b \) if and only if \( A(a \land b) = A(a \land b) \land A(b \land a) \geq 0.5 \).

**Proof.** Let \( A_a = A_b \), then we have \( A_a = A_b \), thus \( A(a \land b) \geq 0.5 \), \( A(b \land a) \geq 0.5 \), that is \( A(a \land b) = A(b \land a) \geq 0.5 \). Conversely, suppose that \( A(a \land b) \land A(b \land a) \geq 0.5 \). For all \( x \in X \), since

\[
(a \land b) \land (b \land a) \geq 0.5.
\]

It follows from Proposition 1 that

\[
A(a \land b) \geq A(b \land a) \land A(a \land b) \geq 0.5
\]

Hence

\[
A(x) = A(a \land b) \land A(a \land b) \geq 0.5
\]

that is \( A_a = A_b \). Similarly, for all \( x \in X \), since

\[
(b \land a) \land (a \land b) \geq 0.5.
\]

It follows from Proposition 1 that

\[
A(b \land a) \geq A(a \land b) \land A(a \land b) \geq 0.5
\]

Hence

\[
A(x) = A(b \land a) \land A(a \land b) \geq 0.5
\]

that is \( A_a = A_b \). Therefore, \( A_a = A_b \). We complete the proof.

**Proposition 9.** Let \( A_a, A_b \in X/A \), then \( A_{ab} = A_{ab} \).

**Proof.** Since

\[
((a \land b) \land (a \land b)) \land (a \land b) = (a \land b) \land (a \land b) \land (a \land b) \land (a \land b) \leq (a \land b) \land (a \land b) \land (a \land b) \land (a \land b) \leq b \land a.
\]

Hence

\[
A((a \land b) \land (a \land b)) \geq A(a \land b) \land A(a \land b) \land (a \land b) \land (a \land b) \geq 0.5
\]

Therefore

\[
A((a \land b) \land (a \land b)) \geq A(a \land b) \land A(a \land b) \land (a \land b) \land (a \land b) \geq 0.5
\]

it follows from Proposition 8 that \( A_{ab} = A_{ab} \). We completed the proof.
Let \( A \) be an \( M_{(e,e\vee q)} \)-fuzzy ideal of \( X \). The operation "\(*" of \( R/A \) is defined as: \( \forall A_x, A_y \in R/A, A_x \ast A_y = A_{x \ast y} \). By Proposition 8, the above operation is reasonable.

**Proposition 10.** \( A \) is an \( M_{(e,e \vee q)} \)-fuzzy ideal of \( X \), then \( R/A = \{R/A; A_y\} \) is an \( M - \)BCI-algebra.

**Proof.** For all \( A_x, A_y, A_z \in R/A \), we have

\[
\left( (A_x \ast A_y) \ast (A_y \ast A_z) \right) = (A_x \ast A_z) = A_{x \ast z};
\]

\[
(A_y \ast (A_x \ast A_z)) = A_{z \ast y} = A_{z \ast y};
\]

\[
A_x \ast A_z = A_{x \ast z};
\]

if \( A_x \ast A_y = A_y, A_y \ast A_z = A_z \), then \( A_x \ast y = A_y, A_{y \ast z} = A_z \), it follows from Proposition 8 that \( A(x \ast y) = A(0), A(y \ast x) = A(0) \), hence \( A(x \ast y) = A(y \ast x) \ast 0.5 = A(0) \ast 0.5 \), then \( A_x = A_y \). Therefore \( R/A = \{R/A; A_y\} \) is a BCI-algebra. For all \( A_x, A_y, A_z, A_w \in M \), define \( A_m = A_w \). Firstly, we verify that \( A_m = A_w \) is reasonable. If \( A_x = A_y \), then we verify that \( A_m = A_w \), that is to verify \( A_{mx} = A_{wy} \). We have

\[
\mu(A_{mx} \ast A_{wy}) = \mu(A_{mx} \ast A_{wy}) \leq \mu(A_y) \ast 0.5 = A_{wy} \ast 0.5,
\]

so we have

\[
A(mx \ast my) \ast A(my \ast mx) \ast 0.5 \geq A(x \ast y) \ast A(y \ast x) \ast 0.5 = A(0) \ast 0.5,
\]

then \( A(mx \ast my) \ast A(my \ast mx) \ast 0.5 = A(0) \ast 0.5 \) that is \( A_{mx} = A_{wy} \).

In addition, for all \( m \in M, A_x, A_y \in R/A \), we have

\[
m(A_x \ast A_y) = m(A_{mx} \ast A_{wy}) = A_{mx} \ast A_{wy},
\]

Therefore \( R/A = \{R/A; A_y\} \) is an \( M - \)BCI-algebra.

**Definition 15.** Let \( \mu \) be an \( M_{(e,e \vee q)} \)-fuzzy subalgebra of \( X \), and \( A \) be an \( M_{(e,e \vee q)} \)-fuzzy ideal of \( X \), we define a fuzzy set of \( X/A \) as follows:

\[
\mu\left( X/A \right) = \sup_{A \in X/A} \mu(x) \ast 0.5, \forall A \in X/A.
\]

**Proposition 11.** \( \mu\left( X/A \right) \) is an \( M_{(e,e \vee q)} \)-fuzzy subalgebra of \( X/A \).

**Proof.** For all \( A_x, A_y \in X/A \), we have

\[
\mu\left( X/A \right) = \sup_{A \in X/A} \mu(x) \ast 0.5,
\]

then we have

\[
A \ast B \left( x, x \right) \geq A \ast B \left( x, x \ast x, y, y \right) \ast A \ast B \left( y, y \right) \ast 0.5;
\]

(3) For all \( (x, y) \in X \times X \), we have

\[
A \ast B \left( x, y \right) \ast 0.5 = A \ast B \left( y, x, y \right) \ast 0.5
\]

then we have
\[ A \times B(m(x, y)) \geq A \times B(x, y) \land 0.5, \forall (x, y) \in X \times X. \]

Therefore \( A \times B \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \times X \).

**Proposition 13.** Suppose \( A \) and \( B \) are fuzzy sets of \( X \), if \( A \times B \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \times X \), then \( A \) or \( B \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \).

**Proof.** Suppose \( A \) and \( B \) are \( M-(e, e \lor q) \)-fuzzy ideals of \( X \), then for all \((x_i, x_j), (y_i, y_j) \in X \times X\), we have

\[
A \times B(x_i, x_j) \geq A \times B((x_i, x_j) \ast (y_i, y_j)) \land A \times B(y_i, y_j) \land 0.5
\]

if \( x_i = y_i = 0 \), then

\[
A \times B(0, x_j) \geq A \times B(0, y_j \ast A \times B(0, y_j) \land 0.5,
\]

then we have

\[
A \times B(0, x) = A(0) \land B(x) = B(x),
\]

thus \( b(x_j) \geq b(x_j \ast y_j) \land 0.5 \). If \( A \times B \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \), then

\[
A \times B(m(x, y)) \geq A \times B(x, y) \land 0.5, \forall (x, y) \in X \times X,
\]

let \( x = 0 \), then

\[
A \times B(m(x, y)) = A \times B(mx, my) = A(mx) \land B(my) = B(my)
\]

\[
\geq A(x) \land B(y) \land 0.5 = A(0) \land B(y) \land 0.5
\]

\[
= B(y) \land 0.5,
\]

then we have

\[
B(my) \geq B(y) \land 0.5, \forall y \in X, m \in M.
\]

Therefore \( B \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \).

**Proposition 14.** If \( B \) is a fuzzy set, \( A \) is a strong fuzzy relation \( A_g \) of \( B \), then \( B \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \) if and only if \( A_g \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \times X \).

**Proof.** If \( B \) is an \( M-(e, e \lor q) \)-fuzzy ideals of \( X \), then for all \((x, y) \in X \times X\), we have

\[
A_g(0, 0) = B(0) \land B(0) \geq B(x) \land 0.5 \land B(y) \land 0.5
\]

\[
= A_g(x, y) \land 0.5;
\]

for all \((x_i, x_j), (y_i, y_j) \in X \times X\), we have

\[
A_g(x_i, x_j) = B(x_i) \land B(x_j)
\]

\[
\geq (B(x_i \ast x_j) \land B(y_j) \land 0.5) \land (B(x_j \ast y_j) \land B(y_j) \land 0.5)
\]

\[
= (B(x_i \ast y_j) \land B(x_j \ast y_j) \land B(y_j) \land B(y_j) \land 0.5)
\]

\[
= A_g(x_i \ast y_j, x_j \ast y_j) \land A_g(y_i, y_j) \land 0.5
\]

\[
= A_g((x_i, x_j) \ast (y_i, y_j)) \land A_g(y_i, y_j) \land 0.5;
\]

for all \((x, y) \in X \times X\), we have

\[
A_g(m(x, y)) = A_g(mx, my) = B(mx) \land B(my)
\]

\[
\geq B(x) \land 0.5 \land B(y) \land 0.5 = A_g(x, y) \land 0.5.
\]

Therefore \( A_g \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \times X \). Conversely, suppose \( A_g \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \times X \), then for all \((x_i, x_j), (y_i, y_j) \in X \times X\), we have

\[
B(0) \land B(0) \geq A_g(x_i, x_j) \land B(x) \land B(y) \land 0.5,
\]

for all \((x, y) \in X \times X\), we have

\[
B(x) \land B(x) \geq (B(x) \land B(y_j)) \land B(0) \land 0.5,
\]

\[
= B(x) \land B(0) \land 0.5,
\]

if \( A_g \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \times X \), then

\[
A_g(m(x, y)) \geq A_g(x, y), \forall x, y \in X \times X, m \in M.
\]

We have

\[
B(mx) \land B(my) = A_g(mx, my) \geq A_g(x, y) \land 0.5 = B(x) \land B(y) \land 0.5.
\]

if \( x = 0 \), then

\[
B(0) \land B(0) = A_g(0, 0) \geq A_g(0, y) \land 0.5 = B(0) \land B(y) \land 0.5,
\]

namely, \( B(my) \geq B(y) \land 0.5 \). Therefore \( B \) is an \( M-(e, e \lor q) \)-fuzzy ideal of \( X \).

**REFERENCES**