On the fuzzy difference equation

\[ x_{n+1} = A + \sum_{i=0}^{k} \frac{B_i}{x_{n-i}} \]

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Abstract—In this paper, we study the existence, the boundedness and the asymptotic behavior of the positive solutions of a fuzzy nonlinear difference equation

\[ x_{n+1} = A + \sum_{i=0}^{k} \frac{B_i}{x_{n-i}} \quad n = 0, 1, \cdots \]

where \((x_n)\) is a sequence of positive fuzzy numbers, \(A, B_i\) and the initial values \(x_{-k}, x_{-k+1}, \cdots, x_0\) are positive fuzzy numbers, \(k \in \{0, 1, 2, \cdots\}\).

We need the following definitions: A is said to be a fuzzy number if \(A : R \rightarrow [0, 1] \) satisfies the below (i)-(iv)

(i) \(A\) is normal, i.e. there exists an \(x \in R\) such that \(A(x) = 1\);
(ii) \(A\) is fuzzy convex, i.e. for all \(t \in [0, 1]\) and \(x_1, x_2 \in R\) such that

\[ A(tx_1 + (1 - t)x_2) \geq \min\{A(x_1), A(x_2)\} \]

(iii) \(A\) is upper semi-continuous;
(iv) The support of \(A\), \(\supp A = \bigcup_{\alpha \in [0,1]} [A_\alpha] = \{x : A(x) > 0\}\) is compact.

The \(\alpha\)-cuts of \(A\) are denoted by \([A]_\alpha = \{x \in R : A(x) \geq \alpha\}, \alpha \in [0, 1]\); it is clear that the \([A]_\alpha\) are closed interval. We say that a fuzzy number is positive if \(\supp A \subset (0, \infty)\).

It is obvious that if \(A\) is a positive real number then \(A\) is a fuzzy number and \([A]_\alpha = [A, A], \alpha \in [0, 1]\). Then we say that \(A\) is a trivial fuzzy number.

Let \(A, B\) be fuzzy numbers with \([A]_\alpha = [A_l, \alpha, A_r, \alpha], [B]_\alpha = [B_l, \alpha, B_r, \alpha], \alpha \in (0, 1]\). We define a norm on fuzzy numbers space as follows:

\[ ||A|| = \sup_{\alpha \in (0,1]} \max\{|A_l, \alpha|, |A_r, \alpha|\}. \]

We take the following metric:

\[ D(A, B) = \sup_{\alpha \in (0,1]} \max\{|A_l, \alpha - B_l, \alpha|, |A_r, \alpha - B_r, \alpha|\}. \]

The fuzzy analog of the boundedness and persistence (see [9,11]) as follows: we say that a sequence of positive fuzzy numbers \((x_n)\) persists (resp. is bounded) if there exists a positive real number \(M\) (resp. \(N\)) such that

\[ \sup x_n \subset [M, \infty) (\text{resp. } \sup x_n \subset (0, N)], \quad n = 1, 2, \cdots. \]

We say that \(x_n\) is bounded and persists if there exist positive real numbers \(M, N > 0\) such that

\[ \sup x_n \subset [M, N], \quad n = 1, 2, \cdots. \]

We say \((x_n)\), \(n = 1, 2, \cdots\), is an unbounded sequence if the norm \(\|x_n\|, n = 1, 2, \cdots\), is an unbounded sequence.

We say that \(x_n\) is a positive solution of (2) if \((x_n)\) is a sequence of positive fuzzy numbers which satisfies (2). We say a positive fuzzy number \(x\) is a positive equilibrium for (2) if

\[ x = A + \sum_{i=0}^{k} \frac{B_i}{x}. \]
Let \((x_n)\) be a sequence of positive fuzzy numbers and \(x\) is a positive fuzzy number. Suppose that
\[
[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0, 1], \quad n = 0, 1, 2, \ldots, (3)
\]
and
\[
[x]_\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0, 1] \quad (4)
\]
We say that the sequence \((x_n)\) converges to \(x\) with respect to \(D\) as \(n \to \infty\) if \(\lim_{n \to \infty} D(x_n, x) = 0\).

Suppose that (2) has a unique positive equilibrium \(x\). We say that the positive equilibrium \(x\) of (2) is stable if for every \(\varepsilon > 0\) there exists a \(\delta = \delta(\varepsilon) > 0\) such that for every positive solution \(x_n\) of (2), which satisfies \(D(x_n, x) \leq \delta, i = 0, 1, \ldots, k\) we have \(D(x_n, x) \leq \varepsilon\) for all \(n > 0\).

Moreover, we say that the positive equilibrium \(x\) of (2) is asymptotically stable, if it is stable and every positive solution of (2) tends to the positive equilibrium of (2) with respect to \(D\) as \(n \to \infty\).

The purpose of this paper is to study the existence of positive solutions of (2). Furthermore, we give some conditions so that every positive solution of (2) is boundedness and persistence. Finally, under some conditions we prove that (2) has a unique positive equilibrium \(x\) which is asymptotically stable.

II. MAIN RESULTS

Firstly we study the existence of the positive solutions of (1). We need the following lemma which is a slight generalization of Lemma 2.1 of [11].

**Lemma 2.1.** Let \(f : R_+^{2k+3} \to R_+\) be continuous, \(A_i, i = 0, 1, \ldots, 2k + 3\), are fuzzy numbers, Then
\[
[f(A_0, \ldots, A_{2k+3})]_\alpha = f([A_0]_\alpha, \ldots, [A_{2k+3}]_\alpha), \quad \alpha \in (0, 1].
\]

**Theorem 2.1.** Consider equation (2) where \(A, B_i\) are positive fuzzy numbers. Then for any positive fuzzy numbers \(x_{-i}, i = 0, 1, \ldots, k\), there exists a unique positive solution \(x_n\) of (2).

**Proof.** Suppose that there exists a sequence of fuzzy numbers \((x_n)\) satisfying (2) with the initial values \(x_{-k}, x_{-k+1}, \ldots, x_0\). Consider the \(\alpha\)-cuts, \(\alpha \in (0, 1], n = -k, -k + 1, \ldots, \)
\[
\begin{align*}
[x_n]_\alpha &= [L_{n,\alpha}, R_{n,\alpha}], [A]_\alpha = [A_0, A_r, \alpha], \\
[B_i]_\alpha &= [B_i, l, A, i = 0, 1, \ldots, k]
\end{align*}
\]
(5)

Then from (2), (5) and Lemma 2.1 it follows that
\[
[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[ A + \sum_{i=0}^{k} B_{i, l, A} \right]_\alpha \\
= [A]_\alpha + \sum_{i=0}^{k} \left[ B_{i, l, A} \right]_\alpha \\
= [A_{l, A}]_\alpha + \sum_{i=0}^{k} \left[ B_{i, l, A}, B_{i, r, A} \right]_\alpha \\
= \left[ A_{l, A} + \sum_{i=0}^{k} B_{i, l, A}, A_{r, A} + \sum_{i=0}^{k} B_{i, r, A} \right]_\alpha
\]
from which we have that for \(n = 0, 1, \ldots, \alpha \in (0, 1]\)
\[
\begin{align*}
L_{n+1,\alpha} &= A_{l, A} + \sum_{i=0}^{k} B_{i, l, A} R_{n-i,\alpha}, \\
R_{n+1,\alpha} &= A_{r, A} + \sum_{i=0}^{k} B_{i, r, A} R_{n-i,\alpha},
\end{align*}
\]
(6)

Then it is obvious that for any \((L_{j, A}, R_{j, A}), j = -k, -k + 1, \ldots, 0\), there exists a unique solution \((L_{n,\alpha}, R_{n,\alpha})\) with the initial conditions \((L_{j, A}, R_{j, A}), j = -k, -k + 1, \ldots, 0, \alpha \in (0, 1]\).

Conversely we prove that \([L_{n,\alpha}, R_{n,\alpha}]\), where \((L_{n,\alpha}, R_{n,\alpha})\) is the solution of the system (6) with the initial values \((L_{-i, A}, R_{-i, A}), i = 0, 1, \ldots, k\), determines the solution \(x_n\) of (2) with the initial values \(x_{-k}, x_{-k+1}, \ldots, x_0\) such that (3) holds.

From Theorem 2.1 of [14] and since \(A, B_i, x_{-i}, i = 0, 1, \ldots, k\), are positive fuzzy numbers for any \(\alpha \in (0, 1]\), \(\alpha_1 \leq \alpha_2\), we have
\[
\begin{align*}
0 < A_{r, A_1} &\leq A_{r, A_2} \leq A_{r, A}, \\
0 < B_{i, A_1} &\leq B_{i, A_2} \leq B_{i, A}, \\
0 < L_{-i, A_1} &\leq L_{-i, A_2} \leq R_{-i, A_2} \leq R_{-i, A},
\end{align*}
\]
(7)

We claim that
\[
L_{n,\alpha} \leq L_{n,\alpha_2} \leq R_{n,\alpha_2} \leq R_{n,\alpha}, \quad n = 0, 1, 2, \ldots, (8)
\]

We prove it by induction. It is obvious from (7) that (8) holds true for \(n = 0\). Suppose that (8) are true for \(n = m\). Then from (6) and (7) it follows that
\[
L_{m+1,\alpha} = A_{l, A} + \sum_{i=0}^{k} B_{i, l, A} R_{m-i,\alpha} \leq A_{l, A} + \sum_{i=0}^{k} B_{i, l, A} R_{m-i,\alpha} = L_{m+1,\alpha}
\]
(9)

Therefore (8) are satisfied. Moreover from (6) we have, for \(\alpha \in (0, 1]\),
\[
L_{1,\alpha} = A_{l, A} + \sum_{i=0}^{k} B_{i, l, A} R_{1-i,\alpha}, \quad R_{1,\alpha} = A_{r, A} + \sum_{i=0}^{k} B_{i, r, A} R_{1-i,\alpha}
\]
Since \(A, B, x_{-i}, i = 0, 1, \ldots, k\) are positive fuzzy numbers, then we have that \(A_{l, A}, A_{r, A}, B_{i, A}, B_{i, r, A}, L_{-i, A}, R_{-i, A}\) are left continuous. So from (9) we have that \(L_{1,\alpha}, R_{1,\alpha}\) are also left continuous. By induction we can get that \(L_{n,\alpha}, R_{n,\alpha}, n = 1, 2, \ldots\) are left continuous.
We prove now that the support of \( x_n, \text{supp} x_n = \bigcup_{\alpha \in (0,1]} [I_{\alpha,0}, R_{\alpha,0}] \) is compact. It is sufficient to prove that \( \bigcup_{\alpha \in (0,1]} [I_{\alpha,0}, R_{\alpha,0}] \) is bounded. Let \( n = 1 \). Since \( A, B, x_{i-1}, n = 1, \ldots, k \), are positive fuzzy numbers there exist positive real numbers \( K, L, M_i, N_i, P_i, Q_i, \) \( i = 0, 1, \ldots, k \) such that for all \( \alpha \in (0,1] \)

\[
\begin{align*}
A_{\alpha, i} & \subseteq [K, L], \quad B_{\alpha, i} \subseteq [M_i, N_i], \\
L_{\alpha, i} & \subseteq [P_i, Q_i].
\end{align*}
\]

Therefore from (9) and (10) it can follow easily that

\[
[L_{\alpha,1}, R_{\alpha,1}] \subseteq \left[ K + \frac{\sum_{i=0}^{k} M_i}{Q_i}, L + \frac{\sum_{i=0}^{k} N_i}{P_i} \right], \quad \alpha \in (0,1]
\]

from which it is obvious that, \( \alpha \in (0,1] \),

\[
\bigcup_{\alpha \in (0,1]} [L_{\alpha,0}, R_{\alpha,0}] \subseteq \left[ K + \frac{\sum_{i=0}^{k} M_i}{Q_i}, L + \frac{\sum_{i=0}^{k} N_i}{P_i} \right]. \quad \alpha \in (0,1]
\]

Relation (11) implies that \( \bigcup_{\alpha \in (0,1]} [L_{\alpha,0}, R_{\alpha,0}] \) is compact and \( \bigcup_{\alpha \in (0,1]} [L_{\alpha,0}, R_{\alpha,0}] \subseteq \left( 0, \infty \right) \). Working inductively we can easily prove that \( \bigcup_{\alpha \in (0,1]} [N_{\alpha,0}, R_{\alpha,0}] \) is compact and

\[
\bigcup_{\alpha \in (0,1]} [N_{\alpha,0}, R_{\alpha,0}] \subseteq \left( 0, \infty \right), \quad n = 1, 2, \ldots
\]

Therefore from Theorem 2.1 of [14], relations (8), (12) and since \( L_{n,0}, R_{n,0} \) are left continuous we have that \( [L_{n,0}, R_{n,0}] \) determines a sequence of positive fuzzy numbers \( (x_n) \) such that (3) holds.

We prove now that \( x_n \) is the solution of (2) with the initial conditions \( x_{i-1}, i = 0, 1, \ldots, k \). Since for all \( \alpha \in (0,1] \)

\[
[x_{n+1}]_\alpha = [L_{\alpha+1,0}, R_{\alpha+1,0}]
\]

\[
= \left[ A_{\alpha,0} + \frac{\sum_{i=0}^{k} B_i}{R_{\alpha,0} - L_{\alpha,0}}, A_{\alpha,0} + \frac{\sum_{i=0}^{k} B_i}{R_{\alpha,0} - L_{\alpha,0}} \right]
\]

It follows that \( x_n \) is the solution of (2) with the initial conditions \( x_{i-1}, i = 0, 1, \ldots, k \).

Suppose that there exists another solution \( \bar{x}_n \) of (2) with the initial conditions \( x_{i-1}, i = 0, 1, \ldots, k \). Then arguing as above we can easily prove that

\[
[x_n]_\alpha = [L_{\alpha,0}, R_{\alpha,0}], \quad \alpha \in (0,1], \quad n = 0, 1, \ldots
\]

Then from (3) and (13) we have \( [x_n]_\alpha = [\bar{x}_n]_\alpha, \quad \alpha \in (0,1], \quad n = 0, 1, \ldots \) from which it follows that \( x_n = \bar{x}_n, n = 0, 1, \ldots \). Therefore the proof of theorem 2.1 is completed.

In the following theorem we study the boundedness and persistence of the positive solution of (2). We first give a lemma of [10].

**Lemma 2.2** Let \( X, Y \) be fuzzy numbers and \( [X]_\alpha = [X_{\alpha,0}, X_{\alpha,1}], [Y]_\alpha = [Y_{\alpha,0}, Y_{\alpha,1}], \alpha \in (0,1] \) be the \( \alpha \)-cuts of \( X, Y \) respectively. Let \( Z \) be a fuzzy number such that \( [Z]_\alpha = [Z_{\alpha,0}, Z_{\alpha,1}], \alpha \in (0,1] \). Then

\[
\text{MIN}\{X, Y\} = Z, \quad \text{if and only if} \quad \min\{X_{\alpha,0}, Y_{\alpha,0}\} = Z_{\alpha,0}
\]

\[
\max\{X_{\alpha,1}, Y_{\alpha,1}\} = Z_{\alpha,1}
\]

\[
\text{resp.} \quad \max\{X_{\alpha,0}, Y_{\alpha,0}\} = Z_{\alpha,0}
\]

\[
\min\{X_{\alpha,1}, Y_{\alpha,1}\} = Z_{\alpha,1}
\]

\[
\text{Theorem 2.2.} \quad \text{Every positive solution of (2) is bounded and persists, where} \ A, B, i = 0, 1, \ldots, k \text{ are positive fuzzy numbers.}
\]

**Proof.** Let \( x_n \) be a positive solution of (2). Suppose (5) is satisfied. From (6) it is clear that \( n = k + 2, k + 3, \ldots, \alpha \leq L_{\alpha,0}, \quad A_{\alpha,0} \leq R_{\alpha,0}, \quad \alpha \in (0,1]. \)

Then from (14) we get

\[
\min\{L_{\alpha,0}, A_{\alpha,0}\}, \min\{R_{\alpha,0}, A_{\alpha,0}\} = [A_{\alpha,0}, R_{\alpha,0}]
\]

So from (15) and Lemma 2.2 it follows that

\[
\text{MIN}\{x_n, A\} = A, \quad n \geq k + 2.
\]

Moreover relations (6) and (14) imply that \( n = k + 2, k + 3, \ldots, \quad L_{\alpha,0} \leq D_{\alpha,0}, \quad R_{\alpha,0} \leq D_{\alpha,0}, \quad \alpha \in (0,1]. \)

where

\[
\begin{align*}
D_{\alpha,0} &= A_{\alpha,0} + \frac{1}{A_{\alpha,0}} \sum_{i=0}^{k} B_i, \\
D_{\alpha,0} &= A_{\alpha,0} + \frac{1}{A_{\alpha,0}} \sum_{i=0}^{k} B_i
\end{align*}
\]

Using relation (7) for \( 0 < \alpha_1 \leq \alpha_2 \) we get

\[
0 < D_{\alpha_1,0} \leq D_{\alpha_2,0} \leq D_{\alpha_2,0}
\]

From Theorem 2.1 of [14] and (18) we have \( D_{\alpha,0}, R_{\alpha,0} \) are left continuous. Moreover from (18) and (7) we have

\[
[D_{\alpha,0}, D_{\alpha,0}] \subseteq \left[ K + \frac{1}{L} \sum_{i=0}^{k} M_i, L \right] + \frac{1}{L} \sum_{i=0}^{k} N_i
\]

from which it is obvious that \( \bigcup_{\alpha \in (0,1]} [D_{\alpha,0}, D_{\alpha,0}] \) is compact. Hence from Theorem 2.1 of [14], (19) and since \( D_{\alpha,0}, R_{\alpha,0} \) are left continuous there exists a fuzzy number \( D \) such that \( [D]_\alpha = [D_{\alpha,0}, D_{\alpha,0}] \). Using (17) and Lemma 2.2 it follows that

\[
\text{MAX}\{x_n, D\} = D, \quad n \geq k + 2.
\]

Hence from (16) and (20) it follows that every positive solution \( x_n \) of (2) is bounded and persists. The proof is complete.

In the following we study the existence of a unique positive equilibrium \( x \) of (2) which is asymptotically stable. We need the following lemma.

**Lemma 2.3** Consider the following system of difference equations \( n = 0, 1, 2, \ldots \)

\[
y_{n+1} = a + \sum_{i=0}^{k} \frac{p_i}{y_{n-i}}, \quad z_{n+1} = b + \sum_{i=0}^{k} \frac{q_i}{y_{n-i}}
\]

where \( k \in \{0, 1, 2, \ldots\}, \quad y_{-k}, y_{-k+1}, \ldots, y_{-k}, z_{-k}, z_{-k+1}, \ldots \)

are positive constants and \( a, b, p_i, q_i, i = 0, 1, \ldots, k \) are
positive real numbers. Then the following statements are true:

(i) Every positive solution \((y_n, z_n)\) of (21) satisfies \(n \geq k+2\),
\[
a \leq y_n \leq a + \frac{1}{b} \sum_{i=0}^{k} p_i, \quad b \leq z_n \leq b + \frac{1}{a} \sum_{i=0}^{k} q_i.
\] (22)

(ii) System (21) has a unique positive equilibrium \((y, z)\) given by
\[
y = \frac{ab - \sum_{i=0}^{k} (q_i - p_i)}{2b} + \sqrt{\left(\frac{\sum_{i=0}^{k} (q_i - p_i) - ab}{2b}\right)^2 + 4ab \sum_{i=0}^{k} q_i},
\]
\[
z = \frac{ab - \sum_{i=0}^{k} (p_i - q_i)}{2a} + \sqrt{\left(\frac{\sum_{i=0}^{k} (p_i - q_i) - ab}{2a}\right)^2 + 4ab \sum_{i=0}^{k} p_i}.
\] (23)

(iii) Every positive solution of system (21) converges to the positive equilibrium \((y, z)\) of (21) as \(n \to \infty\).

Proof. (i) Let \((y_n, z_n)\) be a positive solution of (21), from (21) it is obvious that
\[
a \leq y_n, \quad b \leq z_n, \quad n \geq k + 2.
\] (24)

Moreover from (25) and using (21) we have \(n \geq k + 1\),
\[
y_{n+1} = a + \frac{1}{b} \sum_{i=0}^{k} p_i, \quad \frac{\sum_{i=0}^{k} p_i}{y_n-1} \leq a + \frac{1}{b} \sum_{i=0}^{k} p_i,\]
\[
z_{n+1} = b + \frac{1}{a} \sum_{i=0}^{k} q_i, \quad \frac{\sum_{i=0}^{k} q_i}{z_n-1} \leq b + \frac{1}{a} \sum_{i=0}^{k} q_i.
\] (25)

From (24) and (25) it follows that (22) holds.

(ii) Let \(y, z\) be positive real numbers such that
\[
y = a + \sum_{i=0}^{k} \frac{p_i}{y}, \quad z = b + \sum_{i=0}^{k} \frac{q_i}{z}.
\] (26)

Then from (26) we have that the positive real numbers \(y, z\) are given by (23).

(iii) From (22) we have
\[
\lim_{n \to \infty} \sup y_n = L_1 < \infty, \quad \lim_{n \to \infty} \inf y_n = l_1 > 0,
\]
\[
\lim_{n \to \infty} \sup z_n = L_2 < \infty, \quad \lim_{n \to \infty} \inf z_n = l_1 > 0.
\] (27)

From (21) and (27) we have
\[
L_1 \leq a + \frac{1}{l_1} \sum_{i=0}^{k} p_i, \quad l_1 \geq a + \frac{1}{L_2} \sum_{i=0}^{k} p_i,
\]
\[
L_2 \leq b + \frac{1}{l_2} \sum_{i=0}^{k} q_i, \quad l_2 \geq b + \frac{1}{L_1} \sum_{i=0}^{k} q_i.
\]

From which it follows that
\[
bL_1 + \sum_{i=0}^{k} q_i \leq aL_2 + \sum_{i=0}^{k} p_i, \quad aL_2 + \sum_{i=0}^{k} p_i \leq bL_1 + \sum_{i=0}^{k} q_i.
\] (28)

Then relation (28) implies that \(bL_1 + aL_2 \leq aL_2 + bL_1\), from which it follows that
\[
bL_1 + aL_2 = aL_2 + bL_1.
\] (29)

We claim that
\[
L_1 = l_1, \quad L_2 = l_2.
\] (30)

Suppose on contrary that \(l_1 < L_1\). Then from (29) it follows that \(bL_1 + aL_2 = aL_2 + bL_1\) and so \(L_2 < l_2\) which is a contradiction. Hence \(L_1 = l_1\). Similarly we can prove that \(L_2 = l_2\). Therefore (30) are true. Hence from (21) and (30) there exist the limit \(y_n\) and \(z_n\) as \(n \to \infty\) such that
\[
\lim_{n \to \infty} y_n = y, \quad \lim_{n \to \infty} z_n = z
\]

where \((y, z)\) is the unique positive equilibrium of (21). The proof is completed.

Theorem 2.3 Consider Eq.(2) where \(A,B,i=0,1,\cdots k\) are positive fuzzy numbers. Suppose that
\[
\frac{L^2 + \sum_{i=0}^{k} (N_i - M_i)}{2R}
\]
\[
+ \sqrt{\left(\frac{\sum_{i=0}^{k} (M_i - N_i) - L^2}{2R}\right)^2 + 4L^2 B_{\alpha}} \leq \left(\sum_{i=0}^{k} M_i\right)^{\frac{1}{2}}
\] (31)

Then the following statements are true

(i) Eq.(2) has a unique positive equilibrium \(x\).

(ii) The unique equilibrium \(x\) is asymptotically stable.

Proof. (i) Consider the following system, for \(\alpha \in (0,1]\),
\[
L_\alpha = A_{l,\alpha} + \frac{1}{R_{\alpha}} \sum_{i=0}^{k} B_{i,l,\alpha},
\]
\[
R_\alpha = A_{r,\alpha} + \frac{1}{L_{\alpha}} \sum_{i=0}^{k} B_{i,r,\alpha},
\] (32)

Then the positive solution \((L_\alpha, R_\alpha)\) of (32) is given by
\[
L_\alpha = A_{l,\alpha} - \sum_{i=0}^{k} B_{i,r,\alpha} - A_{l,\alpha} - A_{r,\alpha} - \sum_{i=0}^{k} B_{i,l,\alpha} + \sqrt{\left(\frac{\sum_{i=0}^{k} (B_{i,r,\alpha} - B_{i,l,\alpha} - A_{l,\alpha} - A_{r,\alpha})^2 + 4A_{l,\alpha} A_{r,\alpha} \sum_{i=0}^{k} B_{i,r,\alpha}}{2A_{l,\alpha}}\right)}
\]
\[
R_\alpha = A_{l,\alpha} - \sum_{i=0}^{k} B_{i,r,\alpha} - A_{l,\alpha} - A_{r,\alpha} - \sum_{i=0}^{k} B_{i,l,\alpha} + \sqrt{\left(\frac{\sum_{i=0}^{k} (B_{i,l,\alpha} - B_{i,r,\alpha} - A_{l,\alpha} - A_{r,\alpha})^2 + 4A_{l,\alpha} A_{r,\alpha} \sum_{i=0}^{k} B_{i,l,\alpha}}{2A_{r,\alpha}}\right)}
\] (33)

Let \(x_n\) be a positive solution of (2) such that \([x_n]_{\alpha} = [L_{\alpha}, R_{\alpha}], \alpha \in (0,1], n = 0, 1, \cdots\). Then using Lemma 2.3 to the system (6) we have
\[
\lim_{n \to \infty} L_{\alpha,n} = L_{\alpha}, \quad \lim_{n \to \infty} R_{\alpha,n} = R_{\alpha}.
\] (34)

Hence relations (7) and (34), for \(0 < \alpha_1 \leq \alpha_2 \leq 1\), imply that
\[
0 < L_{\alpha_1} \leq L_{\alpha_2} \leq R_{\alpha_2} \leq R_{\alpha_1}.
\] (35)

Since \(A_{l,\alpha}, A_{r,\alpha}, B_{i,l,\alpha}, B_{i,l,\alpha}, i = 0, 1, \cdots, k\) are left continuous from (33) it follows that \(L_{\alpha, R_{\alpha}}\) are also left continuous. Moreover from (33) and (10) we get
\[
R_{\alpha} \leq d = \frac{L^2 + \sum_{i=0}^{k} (N_i - M_i)}{2R} + \sqrt{\left(\frac{\sum_{i=0}^{k} (M_i - N_i) - L^2}{2R}\right)^2 + 4L^2 B_{\alpha}}\]
(36)
Then from (10), (32) and (36) we have
\[ L_\alpha \geq c = K + \sum_{i=1}^{k} M_i. \]  
(37)
Hence relations (36) and (37) imply that \([L_\alpha, R_\alpha] \subset [c, d]\) and so \(\bigcup_{\alpha \in (0,1]} [L_\alpha, R_\alpha] \subset [c, d]\). From which it is clear that
\[ \bigcup_{\alpha \in (0,1]} [L_\alpha, R_\alpha] \text{ is compact and } \bigcup_{\alpha \in (0,1]} [L_\alpha, R_\alpha] \subset (0, \infty) \]  
(38)
So from Theorem 2.1 of [14], relations (35), (38), (5), and (32) and since \(L_\alpha, R_\alpha\) are left continuous we have that \([L_\alpha, R_\alpha], \alpha \in (0, 1]\) determines a fuzzy number \(x\) such that
\[ x = A + \frac{1}{x} \sum_{i=0}^{k} B_i, \quad [x]_\alpha = [L_\alpha, R_\alpha], \alpha \in (0, 1] \]
and so \(x\) is a positive equilibrium of (2).
Suppose there exists another positive equilibrium \(\pi\) of (2). Then there exist functions \(\overline{L}_\alpha : (0, 1] \to (0, \infty), \overline{R}_\alpha : (0, 1) \to (0, \infty)\) such that
\[ \overline{x} = A + \frac{1}{x} \sum_{i=0}^{k} B_i, \quad [\overline{x}]_\alpha = [\overline{L}_\alpha, \overline{R}_\alpha], \alpha \in (0, 1]. \]  
(39)
From (39) it follows that
\[ \overline{L}_\alpha = A_{l,\alpha} + \frac{1}{R_\alpha} \sum_{i=0}^{k} B_{l,i,\alpha}, \quad \overline{R}_\alpha = A_{r,\alpha} + \frac{1}{L_\alpha} \sum_{i=0}^{k} B_{r,i,\alpha} \]
and so \(L_\alpha = \overline{L}_\alpha, R_\alpha = \overline{R}_\alpha, \alpha \in (0, 1]\). Therefore \(x = \pi\). This completes part (i).
(ii) From (34) we have
\[ \lim_{n \to \infty} D(x_n, x) = \lim_{n \to \infty} \sup_{\alpha \in (0, 1]} \{\max\{[L_n - L_\alpha], [R_n - R_\alpha]\}\} \]
(40)
Let \(\varepsilon\) be a positive real number, we consider the positive real number \(\delta\) as follows
\[ \delta < \min\{\varepsilon, c, c + K - d\} \]
(42)
where \(c, d\) are defined in (36) and (37).
Let \(x_n\) be a positive solution of (2) such that
\[ D(x_{n+1}, x) \leq \delta < \varepsilon, \quad i = 0, 1, 2, \ldots, k. \]
(43)
From (43) it follows that \(i = 0, 1, \ldots, k, \hspace{1cm} [L_{n-1,\alpha} - L_\alpha] \leq \delta, \hspace{1cm} [R_{n-1,\alpha} - R_\alpha] \leq \delta, \hspace{0.5cm} \alpha \in (0, 1]\)  
(44)
From (6), (7), (32) and (44) we have
\[ L_{1,\alpha} - L_\alpha = A_{l,\alpha} + \frac{1}{R_\alpha} \sum_{i=0}^{k} B_{l,i,\alpha} - L_\alpha \]
\[ \leq A_{l,\alpha} + \frac{1}{R_\alpha - \delta} \sum_{i=0}^{k} B_{l,i,\alpha} - L_\alpha \]
\[ \leq \frac{L_0 - A_{l,\alpha}}{R_\alpha - \delta} \leq \frac{R_\alpha - A_{l,\alpha}}{R_\alpha - \delta} \]
\[ \leq \frac{R_\alpha - K}{R_\alpha - \delta} \]  
(45)
from (42) and (45) we get
\[ |L_{1,\alpha} - L_\alpha| < \delta < \varepsilon, \hspace{0.5cm} \alpha \in (0, 1]. \]  
(46)
Moreover from (6), (7), (32) and (44) we have
\[ R_{1,\alpha} - R_\alpha = A_{r,\alpha} + \frac{1}{L_\alpha} \sum_{i=0}^{k} B_{r,i,\alpha} - R_\alpha \]
\[ \leq \frac{L_0 - A_{r,\alpha}}{L_\alpha - \delta} \leq \frac{R_\alpha - A_{r,\alpha}}{L_\alpha - \delta} \]
\[ \leq \frac{R_\alpha - K}{L_\alpha - \delta} \]  
(47)
From (31), (42) and (47) we get
\[ |R_{1,\alpha} - R_\alpha| < \varepsilon, \hspace{0.5cm} \alpha \in (0, 1] \]  
(48)
From (47) and (48), working inductively we can easily prove that
\[ |L_n - L_\alpha| < \varepsilon, \hspace{0.5cm} |R_n - R_\alpha| < \varepsilon, \hspace{0.5cm} \alpha \in (0, 1] \]
(49)
and so \(D(x_n, x) < \varepsilon, n \geq 0\). Therefore the positive equilibrium \(x\) is stable, and noting (41). So the positive equilibrium \(x\) is asymptotically stable. The proof is complete.

### III. Conclusion

In this paper, we study the existence of positive solution to fuzzy difference equation \(x_{n+1} = A + \sum_{i=0}^{k} \frac{B_{i}x_{n-i}}{x_{n-i}}, \hspace{1cm} n = 0, 1, \ldots, \). Under certain conditions, we prove that the positive solutions are bounded and persists. Furthermore, we prove that the equation has a unique positive equilibrium which is asymptotically stable.

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### References

[3] R. DeVault, G. Ladas, S. W. Schultz, Necessary and sufficient conditions the boundedness of \(x_{n+1} = A/x_{n} + B/x_{n-1}^{q}, \hspace{1cm} n = 0, 1, \ldots, \) J. Difference Equations Appl. 3(1998)259-266.
[9] G. Papaschinopoulos, C. J. Schinas, On the fuzzy difference equation \(x_{n+1} = \sum_{i=0}^{k} A_{i}x_{n-i}^{q} + 1/x_{n-k}^{p}, \hspace{1cm} n = 0, 1, \ldots, \) J. Difference Equation Appl. 6(2000)75-89.
[10] G. Papaschinopoulos, B. K. Papadopoulos, On the fuzzy difference equation \(x_{n+1} = A + B/x_{n}^{q}, \hspace{1cm} n = 0, 1, \ldots, \) Soft Comput. 6(2002)456-461.
[11] G. Papaschinopoulos, B. K. Papadopoulos, On the fuzzy difference equation \(x_{n+1} = A + x_{n}/x_{n-m}, \hspace{1cm} n = 0, 1, \ldots, \) Fuzzy Sets and Systems 129(2002)73-81.


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