A Class of Recurrent Sequences Exhibiting Some Exciting Properties of Balancing Numbers

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Abstract—The balancing numbers are natural numbers \( n \) satisfying the Diophantine equation \( 1 + 2 + 3 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r); r \) is the balancer corresponding to the balancing number \( n \). The \( n \)^{th} balancing number is denoted by \( B_n \) and the sequence \( \{B_n\}_{n=1}^{\infty} \) satisfies the recurrence relation \( B_{n+1} = 6B_n - B_{n-1} \). The balancing numbers possess some curious properties, some like Fibonacci numbers and some others are more interesting. This paper is a study of recurrent sequence \( \{x_n\}_{n=1}^{\infty} \) satisfying the recurrence relation \( x_{n+1} = Ax_n - Bx_{n-1} \) and possessing some curious properties like the balancing numbers.

Keywords—Recurrent sequences, Balancing numbers, Lucas balancing numbers, Binet form.

I. INTRODUCTION

The balancing numbers originally introduced by Behera and Panda [1] are natural numbers \( n \) satisfying the Diophantine equation \( 1 + 2 + 3 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r) \), where \( r \) is called the balancer corresponding to the balancing number \( n \). It is proved in [1] (see also [3]) that the sequence of balancing numbers \( \{B_n\}_{n=1}^{\infty} \) are solution of the second order linear recurrence \( y_{n+1} = 6y_n - y_{n-1}, y_0 = 0, y_1 = 1 \). The Binet form of this sequence is \( B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \) where \( \lambda_1 = 3 + \sqrt{8} \) and \( \lambda_2 = 3 - \sqrt{8} \). In a subsequent paper Panda [2], unveiled some fascinating properties of balancing numbers. These properties are:

- The sum of first \( n \) odd balancing numbers is equal to the square of the \( n \)\(^{th} \) balancing numbers—a property similar to the fact that the sum of first \( n \) odd natural numbers is equal to \( n^2 \). This property is neither satisfied by the cobalancing numbers [3] nor by the Fibonacci numbers.
- The greatest common divisor of two balancing numbers is a balancing number; in particular, the greatest common divisor of \( B_m \) and \( B_n \) is \( B_k \), where \( k \) is the greatest common divisor of \( m \) and \( n \). This property is true for Fibonacci numbers also.
- \( B_{m+n} = B_mC_n + C_mB_n \) a property similar to \( \sin(x + y) = \sin x \cos y + \cos x \sin y \), where \( C_n = \sqrt{8B_n^2 + 1} \) is a sequence whose terms are known as Lucas balancing numbers and satisfy a recurrence relation identical with balancing numbers.

II. RESULTS

We consider a class of recurrent second order sequences \( x_{n+1} = Ax_n - Bx_{n-1} \), \( x_0 = 0, x_1 = 1 \) such that \( A^2 - 4B > 0 \) and study conditions under which these sequences would satisfy some of the fascinating properties of balancing numbers mentioned in the last paragraph.

Let us start with a second order linear recurrence \( x_{n+1} = Ax_n - Bx_{n-1} \), \( x_0 = 0, x_1 = 1 \)

where \( A \) and \( B \) are natural numbers such that \( A^2 - 4B > 0 \).

The auxiliary equation of this recurrence is given by

\[ \alpha^2 - A\alpha + B = 0 \]

which has, because of the condition \( A^2 - 4B > 0 \), the unequal real roots

\[ \alpha_1 = \frac{A + \sqrt{A^2 - 4B}}{2}, \quad \alpha_2 = \frac{A - \sqrt{A^2 - 4B}}{2} \]

The general solution is given by

\[ x_n = P\alpha_1^n + Q\alpha_2^n, \]

and using the initial conditions, we get the Binet form

\[ x_n = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}, \quad n = 0, 1, 2, \ldots \]

To find the conditions under which

\[ x_1 + x_3 + \cdots + x_{2n-1} = x_2^n, \]

it is enough to find conditions for

\[ x_{2n+1} = x_{2n}^2 - x_n^n \]

We note that \( \alpha_1 + \alpha_2 = A \) and \( \alpha_1\alpha_2 = B \) and

\[ x_{2n+1}^2 - x_n^n = \left[ \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} \right]^2 - \left[ \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right]^2 \]

\[ = \frac{\alpha_1^{2n+2} + \alpha_2^{2n+2} - \alpha_1^{2n} - \alpha_2^{2n} - 2B^{n+1} + 2B^n}{(\alpha_1 - \alpha_2)^2}, \]

and

\[ x_{2n+1} = x_{2n}^2 - x_n^n \]

is equivalent to

\[ (\alpha_1 - \alpha_2)(\alpha_1^{2n+1} - \alpha_2^{2n+1}) = \alpha_1^{2n+2} + \alpha_2^{2n+2} - \alpha_1^{2n} - \alpha_2^{2n} - 2B^{n+1} + 2B^n \]

which yields

\[ B(\alpha_1^{2n} + \alpha_2^{2n}) = \alpha_1^{2n} - \alpha_2^{2n} + 2B^{n+1} - 2B^n. \]

Further rearrangement converts the last equation to

\[ (B - 1)(2B^n - (\alpha_1^{2n} + \alpha_2^{2n})) = 0 \]
and applying $\alpha_1\alpha_2 = B$ the last equation finally reduces to

$$(B - 1)(\alpha_1^n - \alpha_2^n) = 0$$

which is possible if $\alpha_1^n = \alpha_2^n$ or $B = 1$. If $\alpha_1^n = \alpha_2^n$, then $\alpha_1 = \alpha_2$ or $\alpha_1 = -\alpha_2$. But $\alpha_1 = \alpha_2$ corresponds to $A^2 - 4B = 0$, which is forbidden by our initial assumption and $\alpha_1 = -\alpha_2$ corresponds to a negative B, which is also forbidden. Thus the only option left for us is $B = 1$.

Conversely, if $B = 1$ then $\alpha_1\alpha_2 = 1$ and

$$x_{n+1}^2 - x_n^2 = \left[\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}\right]^2 - \left[\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right]^2$$

$$= \frac{\alpha_1^{2n+2} + \alpha_2^{2n+2} - \alpha_1^{2n} - \alpha_2^{2n}}{(\alpha_1 - \alpha_2)^2}$$

$$= \frac{\alpha_1^{2n+1}(\alpha_1 - \alpha_2) - \alpha_2^{2n+1}(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)^2}$$

$$= \frac{\alpha_1^{2n+1} - \alpha_2^{2n+1}}{\alpha_1 - \alpha_2}$$

$$= x_{2n+1}$$

leading to

$$x_1 + x_3 + \cdots + x_{2n-1} = x_n^2.$$ 

The above discussion proves the following theorem:

**Theorem 2.1:** Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that $A$ and $B$ are natural numbers satisfying $A^2 - 4B > 0$. Then, for each natural number $n$, a necessary and sufficient conditions for $x_1 + x_3 + \cdots + x_{2n-1} = x_n^2$ to hold is $B = 1$.

The balancing number also satisfies a relation

$$B_2 + B_4 + \cdots + B_{2n} = B_nB_{n+1}.$$ 

We next investigate the conditions under which

$$x_2 + x_4 + \cdots + x_{2n} = x_nx_{n+1}.$$ 

It is enough to find conditions under which

$$x_nx_{n+1} - x_{n-1}x_n = x_{2n}.$$ 

This is equivalent to

$$x_n(x_{n+1} - x_{n-1})$$

$$= \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \left[\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} - \frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2}\right]$$

$$= \frac{\alpha_1^{2n+1} + \alpha_2^{2n+1} - \alpha_1^{2n-1} - \alpha_2^{2n-1} - B^n(\alpha_1 + \alpha_2)}{(\alpha_1 - \alpha_2)^2}$$

$$+ \frac{B^{n-1}(\alpha_1 + \alpha_2)}{(\alpha_1 - \alpha_2)^2}$$

$$= \frac{\alpha_1^{2n} - \alpha_2^{2n}}{\alpha_1 - \alpha_2}.$$ 

On rearrangement we get

$$(\alpha_1 - \alpha_2)(\alpha_1^{2n} - \alpha_2^{2n}) = \alpha_1^{2n+1} + \alpha_2^{2n+1} - \alpha_1^{2n-1} - \alpha_2^{2n-1} - B^n(\alpha_1 + \alpha_2) + B^{n-1}(\alpha_1 + \alpha_2).$$

which leads to

$$(B - 1)(\alpha_1^{2n-1} + \alpha_2^{2n-1}) = B^{n-1}(B - 1)(\alpha_1 + \alpha_2)$$

which is possible for all $n$ if $B = 1$.

Conversely, it can be easily seen that if $B = 1$, then $x_nx_{n+1} - x_{n-1}x_n = x_{2n}$. The above discussion together with Theorem 2.1 proves

**Theorem 2.2:** Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that $A$ and $B$ are natural numbers satisfying $A^2 - 4B > 0$. Then, for each natural number $n$, a necessary and sufficient conditions for $x_2 + x_4 + \cdots + x_{2n} = x_nx_{n+1}$ is $B = 1$.

While the Binet form for balancing numbers is

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$, the Binet form for the Lucas balancing numbers is

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}.$$ 

Thus, if we define a new sequence

$$y_n = \frac{\alpha_1^n + \alpha_2^n}{2},$$

then it is easy to verify that

$$2x_ny_n = x_{2n},$$

a property similar to that of balancing numbers. In addition, we observe that $\alpha_1 - \alpha_2 = \sqrt{A^2 - 4B}$, so that

$$(\alpha_1 - \alpha_2)^2 = A^2 - 4B$$

is a natural number. Thus in all cases where $\sqrt{A^2 - 4B}$ is irrational, we have

$$y_m + \sqrt{A^2 - 4B} = \alpha_1^m,$$

leading to

$$x_{2n}y_n = x_{m+n},$$

Comparing rational and irrational parts from both sides, we get

$$y_{m+n} = y_my_n + \frac{A^2 - 4B}{4}x_{m+n},$$

and

$$x_{m+n} = x_my_n + y_mx_n.$$ 

The above discussion proves

**Theorem 2.3:** Let $x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1$ be a second order linear recurrence such that $A$ and $B$ are natural numbers and $A^2 - 4B$ is non-square and positive. If $y_n$ is defined as $y_n = \frac{\alpha_1^n + \alpha_2^n}{2}$, then for all natural numbers $m$ and $n$ we have

$$y_{m+n} = y_my_n + \frac{A^2 - 4B}{4}x_{m+n},$$

and

$$x_{m+n} = y_mx_n + y_mx_n.$$ 

A well-known connection between balancing and Lucas balancing numbers is

$$C_n^2 = 8B_n^2 + 1.$$
We can except a similar relationship between the sequences \( x_n \) and \( y_n \). Indeed

\[
x_n^2 = \left( \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \right)^2 = \frac{\alpha_1^{2n} + \alpha_2^{2n} - 2B^n}{A^2 - 4B}.
\]

Thus

\[
\frac{(A^2 - 4B)x_n^2}{4} + B^n = \frac{\alpha_1^{2n} + \alpha_2^{2n} + 2B^n}{4} = \left( \frac{\alpha_1^2 + \alpha_2^2}{2} \right)^2 = y_n^2.
\]

Writing \( D = \frac{A^2 - 4B}{4} \), the last equation can be written as

\[
y_n^2 = B^n + Dx_n^2.
\]

The above equation proves

**Theorem 2.4:** Let \( x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1 \) be a second order linear recurrence such that \( A \) and \( B \) are natural numbers and \( A^2 - 4B > 0 \). If \( y_n \) is defined as \( y_n = \frac{\alpha_1^n + \alpha_2^n}{2} \), then \( y_n^2 = B^n + Dx_n^2 \) where \( D = \frac{A^2 - 4B}{4} \).

We now try to find a recurrence relation for \( y_n \). Since \( \alpha_1 \) and \( \alpha_2 \) are roots of the equation

\[
a^2 - Aa + B = 0
\]

it follows that

\[
a_1^2 - Aa_1 + B = 0,
\]

and

\[
a_2^2 - Aa_2 + B = 0.
\]

Multiplying the last two equations by \( \alpha_1^{n-1} \) and \( \alpha_2^{n-1} \) respectively and rearranging, we get

\[
\alpha_1^{n+1} = A\alpha_1^n + B\alpha_1^{n-1},
\]

and

\[
\alpha_2^{n+1} = A\alpha_2^n + B\alpha_2^{n-1}.
\]

Adding the last two equations and dividing by 2 we arrive at

\[
y_{n+1} = Ay_n - By_{n-1}.
\]

It is clear that \( y_0 = 1 \) and \( y_1 = \frac{1}{2} \). This shows that \( y_n \) satisfies a recurrence relation identical with \( x_n \). Further, if \( A \) is even then \( y_n \) is an integer sequence.

**Theorem 2.5:** Let \( x_{n+1} = Ax_n - Bx_{n-1}, x_0 = 0, x_1 = 1 \) be a second order linear recurrence such that \( A \) and \( B \) are natural numbers and \( A^2 - 4B > 0 \). If \( y_n \) is defined as \( y_n = \frac{\alpha_1^n + \alpha_2^n}{2} \), the sequence \( \{ y_n \}_{n=1}^{\infty} \) satisfies the recurrence relation \( y_{n+1} = Ay_n - By_{n-1} \). Further, \( y_n \) is an integer sequence if \( A \) is even.

We now suppose that \( A \) is even and hence \( \{ y_n \}_{n=1}^{\infty} \) an integer sequence and choose \( B = 1 \) so that the greatest common divisor of \( x_n \) and \( y_n \) is 1 for each \( n \). Let \( k \) and \( n \) be two natural numbers such that \( n > 1 \). Then denoting the greatest common divisor of \( a \) and \( b \) by \( \gcd (a, b) \), we have

\[
(\gcd (x_k, x_{nk})) = (\gcd (x_k, x_{k}y_{(n-1)k} + y_{k} x_{(n-1)k})) = (\gcd (x_k, x_{(n-1)k})).
\]

Iterating recursively, we arrive at

\[
(x_k, x_{nk}) = (x_k, x_k) = x_k.
\]

This proves

**Theorem 2.6:** Let \( x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1 \) be a second order linear recurrence such that \( A \) is an even natural number and \( A^2 - 4 \) is positive. If \( m \) and \( n \) are natural numbers and \( m \) divides \( n \) then \( x_m \) divides \( x_n \).

We now look at the converse of this theorem. Assume that \( m \) and \( n \) are natural numbers such that \( x_m \) divides \( x_n \). Then definitely, \( m < n \) and by Euclid’s division algorithm [4], there exist natural numbers \( k \) and \( r \) such that \( n = mk + r, k \geq 1, 0 \leq r < m \).

By Theorem 2.3

\[
x_n = (x_m, x_n) = (x_m, x_{mk+r}) = (x_m, x_{mk}y_r + y_mx_r).
\]

Since \( m \) divides \( mk \), by Theorem 2.6, \( x_m \) divides \( x_{mk} \) and hence the last equation yields

\[
x_m = (x_m, x_r).
\]

Since \( r < m \), this is impossible unless \( r = 0 \). Thus \( n = mk \) showing that \( m \) divides \( n \). This proves

**Theorem 2.7:** Let \( x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1 \) be a second order linear recurrence such that \( A \) is an even natural number and \( A^2 - 4 \) is positive. If \( x_m \) divides \( x_n \), then \( m \) divides \( n \).

Let \( m \) and \( n \) be two natural numbers such that \( k = (m, n) \). Thus \( k \) divides both \( m \) and \( n \). In view of Theorem 2.6, \( x_k \) divides both \( x_m \) and \( x_n \) and hence \( x_k \) divides \( (x_m, x_n) \). Further if \( s > k \) and \( x_s \) divides \( x_m \) and \( x_n \), then by Theorem 2.7, \( s \) divides both \( m \) and \( n \) and consequently, \( s \) divides \( k \) which is a contradiction. Hence if \( k = (m, n) \), then \( k \) is the largest number such that \( x_k \) divides both \( x_m \) and \( x_n \). The discussion of this paragraph may be summarized as follows:

**Theorem 2.8:** Let \( x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1 \) be a second order linear recurrence such that \( A \) is an even natural number and \( A^2 - 4 \) is positive. If \( m \) and \( n \) are natural numbers then \( (x_m, x_n) = (x_m, x_n) \).

**References**


