Positive solutions of second-order singular differential equations in Banach space  
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Abstract—In this paper, by constructing a special set and utilizing fixed point index theory, we study the existence of solution for the boundary value problem of second-order singular differential equations in Banach space, which improved and generalize the result of related paper.

Keywords—Banach space, cone, fixed point index, singular equation.

I. INTRODUCTION

The singular differential equation arises in a variety of applied mathematics and physics, the theory of singular differential equation is emerging as an important area of investigation since it is much richer than the corresponding theory of concerning equation without singular. In recent years, some new results concerning the Dirichlet boundary value problem of singular differential equation have been obtained by a variety of method ([1-5]). In thesis [6-7] the author investigate the singular equation

$$
\begin{cases}
  x''(t) + f(t, x(t)) = \theta \\
  x(0) = x(1) = \theta 
\end{cases}
$$

convergence in abstract space. Denotes $\alpha$ the Kuratowski noncompactness measure in $E$, $\alpha(\cdot)$ and $\alpha_c(\cdot)$ the Kuratowski noncompactness measure in $E$ and $C[J, E]$ respectively.

Lemma 1.1 Suppose $S \subset C[J, E]$ is a bounded and equicontinuous set on $J$, then $\alpha_c(S) = \sup_{t \in J} \alpha(S(t))$, where $S(t) = \{x(t) : x \in S\}$.

Lemma 1.2 Suppose $P$ is a regular and solid cone in Banach space, $P_t = \{x \in P : \|x\| < \tau\}$, $F : P_t \rightarrow P$ is strictly set contraction, and if $\forall x \in \partial P_t$, and $\lambda \geq 0$, $u_0 \in P\{\theta\}$, we have $x - Fx \neq \lambda u_0$ then $i(F_t, P_t, P) = 1$.

Lemma 1.3 Suppose $P$ is a cone in Banach space, $P_R = \{x \in P : \|x\| < R\}$, $F : P_R \rightarrow P$ is a strictly set contraction, $F \in C[0, 1]$, then $i(F_t, P_R, P) = 1$.

Lemma 1.4 Suppose $V = \{x_n\} \subset L[J, E]$, there exist $g \in L[0, 1]$, for all $x_n \in V$, $\|x_n\| \leq g(t)$, a.e.t $\in J$, then $\alpha(\{\int_0^1 x_n(s)ds : n \in N\}) \leq 2 \int_0^1 \alpha(V(s)ds)$.

II. CONCLUSION

For convenience, we list the following assumptions:

(H1) Let $\varphi(t) = c + d \cdot t, \psi(t) = b + at, t \in [0, 1]$. Suppose $f \in C([0, 1] \times P, P)$ with $\|f(t, x)\| \leq k(t)\|q(x)\|$, where $k : (0, 1) \rightarrow (0, +\infty)$ satisfying $\int_0^1 \varphi(s)\psi(s)k(s)ds < +\infty$, and $q \in C[P, P]$.

(H2) Let $\varphi = ac + ad - b - c, G(s, s) = \frac{\varphi(s)\psi(s)}{\rho} q(r_1, R_1) = \sup_{x \in P_{r_1}P_{r_1}} \|q(x)\| < +\infty$. For any $R_1 > r_1 > 0$, suppose $x \in P_{r_1}P_{r_1}$,

$$
\int_0^1 G(s, s)k(s)\frac{\varphi(s)\psi(s)}{\rho + bd} r_1 ds < +\infty, \text{ and there exists } R > 0 \text{ such that } \int_0^1 G(s, s)k(s)\frac{\varphi(s)\psi(s)}{\rho + bd} R ds < R.
$$

(H3) $f(t, x)$ is continuous uniformly on $[\delta, 1 - \delta] \times P_{r_1}P_{r_1}$, where $\delta \in (0, \frac{1}{2})$.

(H4) For all $t \in (0, 1)$ and bounded set $D \subset P_{r_1}P_{r_1}$ there exists an $L$ with $0 \leq L < \frac{2ac}{bd}$ such that $\alpha(f(t, D)) \leq \frac{1}{ac}(D)$.

(H5) $f(t, x)$ has a lower bound $\delta > 1$, $h(t)$ is uniformly continuous with respect to $t \in [0, 1)$, and $0 \leq \int_0^1 G(s, s)h(s)ds < +\infty$, where $P^* = \sup_{x \in P \{\theta\}} |\alpha(x)|$ holds uniformly with $k(t) = \int_0^t f(t, x)dx$.

(H6) There exists $k^* \in P^*, \|k^*\| = 1$ and $[s_1, s_2] \subset [0, 1]$ such that $\lim_{x \rightarrow +\infty} k^*(t) = 0$ holds uniformly with $k^*(t) = \int_0^t f(t, x)dx$. 

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In order to overcome the difficulty caused by singular, we
meanwhile by (5) we can see
\[\{Ax_{m}\} \subset \{x_{m}\}\] such that \(|Ax_{m} - Ax|_{C} \geq \epsilon_{0}(i = 1, 2, ...)\),
since \(\{Ax_{m}\}\) is relatively compact in \([C_{E}]\), the relative
compactness of \(\{Ax_{m}\}\) implies that \(\{Ax_{m}\}\) contains a
subsequence which converges to some \(y \in K\), no loss of
generality we may assume that \(\lim_{m \rightarrow +\infty} Ax_{m} = y\), i.e.
\(\lim_{m \rightarrow +\infty} ||Ax_{m} - y||_{C} = 0\), obviously this is in contradiction
with \(y = Ax, so A is continuous.
On the other hand, by virtue of \((H_{1}) - (H_{2})\) and the
inequality (5), we know \(A\) is bounded from \(K_{r}\) to \(K\).

\textbf{Lemma 2.2} Suppose \((H_{1}) - (H_{2})\) hold, then for \(\forall r > 0\),
\(A_{S}\) is bounded set and equicontinuous on \(J\), by lemma 1.1 we know
\[\alpha_{c}(A(S)) = \sup_{t \in J} \alpha(A(S))\]
where \(AS(t) = \{Ax(t) : x \in S, t \in J\}, let D_{S} = \{\int_{t-\delta}^{t} G(t,s)f(s,x(s))ds : x \in S, \delta \in (0,\frac{1}{2})\}\),
then by virtue of \((H_{1}) - (H_{2})\), for \(x \in S, t \in J\) we have
\[\parallel \int_{t-\delta}^{t} G(t,s)f(s,x(s))ds - \int_{t}^{t+\delta} G(t,s)f(s,x(s))ds \parallel \leq C_{1} \int_{t-\delta}^{t} G(t,s)k(s)ds + \int_{t}^{t+\delta} G(t,s)k(s)ds\]
where \(C_{1} = q[0,1]\), by virtue of (6) and (7), we know the Hausdorff
distance of \(D_{S}\) and \(\{A(S)\}\)
\[d_{H}(D_{S}, AS) \rightarrow 0, \ \delta \rightarrow 0+\]
so
\[\alpha_{c}(AS) = \lim_{\delta \rightarrow 0+} \alpha_{c}(D_{S})\]

Next we estimate \(\alpha(D_{S})\), because \(\int_{t-\delta}^{t} G(t,s)f(s,x(s))ds \in
(1-2\delta)\overline{\parallel G(t,s)f(s,x(s)) : s \in [\delta,1-\delta]\parallel}\), so by \((H_{3}) -
(H_{4})\) and (9.4.11) in therein [8] we obtain
\[\alpha(D_{S}) = \alpha\left(\int_{t-\delta}^{t} G(t,s)f(s,x(s))ds : x \in S\right)\]
\[\leq (1-2\delta)\alpha\left(\overline{\parallel G(t,s)f(s,x(s)) : s \in [\delta,1-\delta]\parallel}\right)\]
\[\leq \alpha\left(\{G(t,s)f(s,x(s)) : s \in [\delta,1-\delta], x \in S\}\right)\]
\[\leq \frac{1}{4ac} \alpha_{c}(\{f(s,x(s)) : s \in [\delta,1-\delta], x \in S\})\]
\[\leq \frac{1}{2} \alpha_{c}(S, I_{\delta}) = [\delta,1-\delta].\]

Note (8), let \(\delta \rightarrow 0\) when \(\alpha_{c}(S) \neq 0\), we have
\(\alpha(A(S)) \leq \frac{1}{2} \alpha_{c}(S) < \alpha_{c}(S), so A is strict set
correction. On the other hand by lemma 2.1, for \(\forall r > 0\), \(A\) is strictly
set contraction from \(K_{r}\) to \(K\).
Now we give the proof of theorem 2.1.
Proof: By virtue of (H2), for \( \forall t \in (0, 1) \) we have
\[
0 < \int_0^1 G(t, s)h(s)ds \leq \int_0^1 G(s, s)h(s)ds, \text{ otherwise } h(s) = 0, \text{a.e. } s \in J.
\]
Chose \( \varepsilon > 0 \) sufficiently small such that
\[
r' = \int_0^1 \int_0^1 G(t, s)(h(s) - \varepsilon')dsdt > 0. \tag{10}
\]
by (H2), there exists \( r'' \in (0, R) \) when \( ||x|| < r'' \), for \( \forall t \in (0, 1) \), we have
\[
h^*(f(t, x(t))) \geq h(s) - \varepsilon'. \tag{11}
\]
In addition, by virtue of (10) and (12), we have
\[
\int_0^1 h^*(x(t))dt \geq \int_0^1 G(t, s)(h(s) - \varepsilon')dsdt = r' > r. \tag{13}
\]
But for \( \forall t \in J \), since \( h^*(x(t)) \leq ||x(t)|| \leq ||x||_e = r \), this is in contradiction with (13). According to lemma 1.2, we have
\[
i(A, K_R, K) = 0. \tag{14}
\]
Next by (H2) we will show \( i(A, K_R, K) = 1 \), by the homotopy invariance of fixed point index, we only need to show: for \( \forall x \in \partial K_R \) and \( \forall \lambda > 1, A \lambda x \neq Ax \).

In fact, if it is false, then there exist \( x \in \partial K_R \) and some \( \lambda \geq 1 \) such that \( A \lambda x = Ax \), then \( x = \frac{1}{\lambda} Ax \), therefore by (2) we know, for \( \forall t \in (0, 1) \), we have
\[
x(t) = \frac{1}{\lambda \rho}(c(1 - t) + d) \int_0^1 (as + b)f(s, x(s))ds + \frac{1}{\lambda \rho}(at + b) \int_0^1 (c(1 - s) + df(s, x(s))ds,
\]
therefore
\[
x'(t) = \frac{1}{\lambda \rho} \int_0^t (-c)(as + b)f(s, x(s))ds + \frac{a}{\rho \lambda} \int_0^1 c(1 - s) + df(s, x(s))ds
\]
\[
\leq \frac{a}{\rho \lambda} \int_0^1 c(1 - s) + df(s, x(s))ds,
\]
( where \( \leq \) is partial order induced by cone. ) So for \( \forall t \in J \), we have
\[
0 \leq x(t) \leq \frac{a}{\rho \lambda} \int_0^1 \int_0^1 [c(1 - s) + df(s, x(s))dsdt + \frac{b}{\rho \lambda} \int_0^1 \int_0^1 [c(1 - s) + df(s, x(s))ds
\]
\[
= \frac{1}{\lambda} \int_0^1 (as + b)c(1 - s) + df(s, x(s))ds
\]
\[
= \frac{1}{\lambda} \int_0^1 G(s, s)f(s, x(s))ds,
\]
so
\[
R = ||x||_e = \max_{t \in J} ||x(t)||
\]
\[
\leq \frac{1}{\lambda} \int_0^1 G(s, s)k(s)(q^{\lambda}(s)\psi(s))ds = \frac{1}{\lambda} \int_0^1 G(s, s)k(s)(q^{\lambda}(s)\psi(s))ds + \frac{bd}{\rho + bd}Rds < R.
\]
This is in contradiction with (H2), so we have
\[
i(A, K_R, K) = 1. \tag{15}
\]
Select \( R' = \max_{t \in J} \frac{[1 + as_1(c + d - cs_2)]}{\rho + bd} \int_0^1 G(t, s)ds \), by (H6), for \( x > N \), there exists \( N > R \) such that \( k^*(f(t, x)) \geq R' ||x|| \), let \( \overline{R} = R + 1 \) then for \( \forall x \in \partial K_R, \lambda \geq 1 \), we have \( x = A \lambda x \neq Ax \). In fact, if there exist \( \lambda \geq 0, x \in \partial K_R \) such that \( x - A \lambda x = \lambda e \) then
\[
\overline{R} \geq k^*(x(t)) \geq k^*(A \lambda x(t))
\]
\[
\geq \int_0^1 G(t, s)k^*(f(s, x(s)))ds
\]
\[
\geq \int_0^1 G(t, s)k^*(f(s, x(s)))ds
\]
\[
\geq \overline{R} \int_0^1 G(t, s)||x(s)||ds
\]
\[
\geq \overline{R} \int_0^1 G(t, s)(\frac{b + as_1(c + d - cs_2)}{\rho + bd})||x||ds > \overline{R}.
\]
This is a contradiction, so by lemma1.2,
\[
i(A, K_R, K) = 0.
\]
Moreover, by (14)(15), we can see
\[
i(A, K_R \setminus \overline{K_R}, K) = i(A, K_R, K) - i(A, K, K) = 1,
\]
\[
i(A, K_R \setminus \overline{K_R}, K) = i(A, K_R, K) - i(A, K_R, K) = -1.
\]
So \( A \) has fixed point \( x \in K_R \setminus \overline{K_R} \) and \( y \in K \setminus \overline{K_R} \) respectively.

Finally we show \( x \neq y \), we only need to show \( A \) has not fixed point in \( \partial K_R \). Otherwise, assume \( z \in \partial K_R \) is a fixed point, so when \( t \in J \), \( z(t) = \int_0^1 G(t, s)(f(s, z(s)))ds \) and
By $(H_1) - (H_2)$ we have

$$R = \max_{t \in J} \|z(t)\| \geq \frac{(b+at)(c+d-et)}{\rho + bd} R.$$

This is a contradiction, and our conclusion follows. □

**Corollary** Suppose conditions $(H_1) - (H_5)$ hold, or conditions $(H_1) - (H_4)$ and $(H_6)$ hold, BVP (1.1) has at least one positive solution.

**Example:** Suppose $E = l^\infty = \{ x = (x_1, x_2, \ldots, x_n, \ldots) : \sup_n |x_n| < +\infty \}$, for $x \in E$, let $\|x\| = \sup_n |x_n|$, then $(E, \|\|)$ is a Banach space, and $P = \{ x \in E : x_n \geq 0, n = 1, 2, \ldots \}$ is a regular cone in $E$, let the regular constant $c = 1$, we consider the following equations in $E$

$$\begin{cases}
-x''(t) = \frac{\cos t}{\sqrt{t(t-1)}} \left( 1 + \frac{1}{n} (tx_{2n} + \ln(1 + x_n)) \right), \\
x_n(0) = x_n(1) = 0, n = 1, 2, \ldots
\end{cases}$$

(16)

Problem (16) can be think as the type of BVP(1), it is equivalent to $x(t) = (x_1(t), x_2(t), \ldots), f(t) = (f_1, f_2, \ldots); f_n(t, x) = \frac{\cos t}{\sqrt{t(t-1)}} \left( 1 + \frac{1}{n} (tx_{2n} + \ln(1 + x_n)) \right)$, we can see $f(t, x)$ is singular at $t = 0, 1$. Now we check $H_1 - H_5$ hold.

Choose $k(t) = \frac{1}{\sqrt{t(t-1)}}$, $q(x) = (q_1(x), q_2(x), \ldots), q_n(x) = 1 + \frac{1}{n} (tx_{2n} + \ln(1 + x_n)) \text{ for } \forall R_1 > r_1 > 0$, it is easy to get

$$q[r_1, R_1] = \sup_{x \in P_{R_1} \setminus P_{r_1}} \|q(x)\| \leq 1 + R_1 + \ln(1 + R_1).$$

Since

$$\int_0^1 \sqrt{s(1-s)} ds = \frac{\pi}{8}, \quad \int_0^1 \frac{ds}{\sqrt{s(1-s)}} = \pi,$$

therefore

$$\int_0^1 s(1-s)k(s)q(s(1-s)r_1, R_1)ds \leq \frac{\pi}{8} (1 + R_1 + \ln(1 + R_1)) < +\infty.$$  

It is easy to see, if $R$ is sufficiently large then

$$\int_0^1 s(1-s)k(s)q(s(1-s)r_1, R_1)ds \leq R.$$

Hence, we choose a convergent subsequence from $\{ f(t, x^{(n)}) \}$ (where $R_1 > r_1 > 0$ is arbitrary), so $H_1$ hold, and it is equivalent to the case $L=0$. Choose $h^* \in P^*$ such that $h^*(x) = x_1$, so $H_2$ hold. To sum up, $H_1 - H_5$ hold, by the Corollary of theorem 2.1, we know problem (16) has at least one positive solution.

**REFERENCES**


