World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:6, No:8, 2012

Bifurcations of a delayed prototype model

Changjin Xu

Abstract—In this paper, a delayed prototype model is studied. Regarding the delay as a bifurcation parameter, we prove that a sequence of Hopf bifurcations will occur at the positive equilibrium when the delay increases. Using the normal form method and center manifold theory, some explicit formulae are worked out for determining the stability and the direction of the bifurcated periodic solutions. Finally, Computer simulations are carried out to explain some mathematical conclusions.

Keywords—Prototype model; Stability; Hopf bifurcation; Delay; Periodic solution.

I. INTRODUCTION

IN 2002, Uçar[1] investigated the following simple nonlinear system with delay element

$$\frac{dx(t)}{dt} = \delta x(t-\tau) - \epsilon[x(t-\tau)]^3, (t \geq t_0), \tag{1}$$

where δ and ϵ are positive parameters; t_0 is the initial interval and $\tau > 0$ corresponds to the delay time in which represents the time interval between the start of an event at one point and its resulting action at another point in the system. Uçar[1] presented the rich dynamical behaviors of system (1) by means of fifth-order Runge-Kutta ordinary differential solver, embedded in Matlab toolboxes. It has been shown that the simple system (1) with a time delay can exhibit very complex behavior include chaos and it can be used as a prototype model for investigating chaotic behaviors in engineering science. In 2003, Uçar[2] further studied the model (1). The effect of time delay on the global behaviors of system (1) had been analyzed with the bifurcation diagram for a range of the time delay. By use of the Euler method and Runge-Kutta discretization, Peng[3] proposed a discrete version of system (1) as follows

$$u(k+1) = u(k)+\alpha(\delta,\tau,n)u(k-n)-\beta(\epsilon,\tau,n)u(k-n)^3, \tag{2}$$

where $\alpha(\delta,\tau,n) = \delta\tau/n, \beta(\epsilon,\tau,n) = \epsilon\tau/n$ and $u(k)$ is an approximate value to $x(kh)$, $h = 1/n$ is a step-size. In [3], efficient computation of Hopf bifurcation, stable limit cycle(periodic solutions), symmetrical breaking bifurcations and chaotic behavior of system (2) was proposed. In order to investigate the effect of parameters of system, Li et al.[4] made an discussion on the Hopf bifurcation of following system

$$\frac{dx(t)}{dt} = ax(t-\tau) - b[x(t-\tau)]^3, \tag{3}$$

which is in fact equivalent to system (1). By choosing the coefficient a as a bifurcation parameter, the local stability and the existence of Hopf bifurcation were considered. Moreover, the stability of bifurcating periodic solutions and the direction of Hopf bifurcation were determined by applying the normal form theory and center manifold theorem.

Based on former work[1-4], we further devote to explore the dynamical behaviors of system (1), i.e., by regarding the delay as bifurcation parameter, we will investigate the natures of Hopf bifurcation of system (1). In recent years, there are a number of papers which deal with this topic(see[5-19]).

This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are studied. In Section 3, the direction of Hopf bifurcation and the stability and periodic of bifurcating periodic solutions on the center manifold are determined. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

II. STABILITY OF THE EQUILIBRIUM AND LOCAL HOPF BIFURCATIONS

Considering the biological interpretations of population, in this paper, we only investigated the positive equilibrium point of system (1). It is obvious that system (1) has a unique positive equilibrium point $x^* = \sqrt{\frac{\delta}{\epsilon}}$.

Let $\bar{x}(t) = x(t) - x^*$, Substituting this into (1) and still denote $\bar{x}(t)$ by $x(t)$, then (1) takes the form

$$\frac{dx(t)}{dt} = -2\delta x(t-\tau) - 3\sqrt{\delta\epsilon}x^2(t-\tau) - \epsilon x^3(t-\tau), \tag{4}$$

Then the linearization of system (4) at the equilibrium (0,0) is given by

$$\frac{dx(t)}{dt} = -2\delta x(t-\tau), \tag{5}$$

whose associated characteristic equation of (5) takes the form

$$\lambda + 2\delta e^{-\lambda\tau} = 0, \tag{6}$$

Let $\lambda = i\omega_0, \tau = \tau_0$, and substituting this into (5). Separating the real and imaginary parts, we get

$$2\delta\cos\omega_0\tau = 0, 2\delta\sin\omega_0\tau = \omega_0. \tag{7}$$

Since $a_1 < 0$, then it is easy to obtain

$$\omega_0 = 2\delta, \tau = \tau_k = \frac{1}{2\delta}\left[k\pi + \frac{\pi}{2}\right], k = 0,1,2,\cdots. \tag{8}$$

Note that when $\tau = 0$, (6) becomes

$$\lambda = -2\delta < 0. \tag{9}$$

The above analysis leads to

Lemma 2.1. *System (1) admits a pair of purely imaginary roots* $\pm i\omega_0$ *when* $\tau = \tau_k, k = 0,1,2,\cdots$.

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of Eq.(6) near $\tau = \tau_k$ satisfying $\alpha(\tau_k) = 0,\ \omega(\tau_k) = \omega_0$. Due to functional

C. Xu is with the Guizhou Key Laboratory of Economics System Simulation, School of Mathematics and Statistics, Guizhou College of Finance and Economics, Guiyang 550004, PR China e-mail: xcj403@126.com.

differential equation theory, for every $\tau_k, k = 0, 1, 2, \cdots$, there exists a $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand side of (6) and taking the derivative of λ with respect to τ, we get

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{1}{2\delta\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

It follows together with (7) that

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]\Big|^{-1}_{\tau=\tau_k} = \mathrm{Re}\left\{\frac{1}{2\delta\lambda e^{-\lambda\tau}}\right\}_{\tau=\tau_k} = \frac{\sin\omega_0\tau_k}{2\delta\omega_0} = \frac{1}{4\delta^2} > 0$$

Thus

$$\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]\Big|_{\tau=\tau_k}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]\Big|^{-1}_{\tau=\tau_k}\right\} > 0.$$

According to the results of Kuang[20] and Hale[21], we have

Theorem 2.1. *The positive equilibrium x^* of system (1) is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau \geq \tau_0$. System (1) underdoes a Hopf bifurcation at the positive equilibrium x^* when $\tau = \tau_k, k = 0, 1, 2, \cdots$.*

III. Direction and stability of the Hopf bifurcation

In the previous section, we have obtained conditions for Hopf bifurcation to occur when $\tau = \tau_k$. In this section, we shall derive the explicit formulae for determining the direction, stability, and period of these periodic solutions bifurcating from the equilibrium x^* at these critical value of τ, by using techniques from normal form and center manifold theory [22], Throughout this section, we always assume that system (4) undergoes Hopf bifurcation at the equilibrium x^* for $\tau = \tau_k$, and then $\pm i\omega_0$ are corresponding purely imaginary roots of the characteristic equation at the equilibrium x^*.

For convenience, let $\bar{x}(t) = x(\tau t), \tau = \tau_k + \mu$ and still denote $\bar{x}(t)$ by $x(t)$, then system (4) can be written as an FDE in $C = C[-1, 0], R)$ as

$$\dot{x}(t) = L_\mu(x_t) + F(\mu, x_t), \tag{10}$$

where $x_t(\theta) = x(t + \theta) \in C$, and $L_\mu : C \to R, F : R \times C \to R$ are given by

$$L_\mu(\phi) = -(\tau_k + \mu)2\delta\phi(-1), \tag{11}$$

and

$$f(\mu, \phi) = (\tau_k + \mu)[-3\sqrt{\delta\epsilon}\phi^2(-1) - \epsilon\phi^3(-1)]. \tag{12}$$

From the discussions in Section 2, we know that if $\mu = 0$, then system (10) undergoes a Hopf bifurcation at the zero equilibrium and the associated characteristic equation of system (10) has a pair of simple imaginary roots $\pm i\omega_0\tau_k$.

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta, \mu)$ in $[-1, 0] \to R$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta). \tag{13}$$

In fact, choosing

$$\eta(\theta, \mu) = 2\delta\delta_1(\theta + 1), \tag{14}$$

where $\delta_1(\theta)$ is Dirac delta function, then (14) is satisfied. For $\varphi \in (C[-1, 0], R)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^{0} d\eta(s, \mu)\phi(s), & \theta = 0 \end{cases} \tag{15}$$

and

$$R\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ f(\mu, \phi), & \theta = 0. \end{cases} \tag{16}$$

Then (1) is equivalent to the abstract differential equation

$$\dot{x_t} = A(\mu)x_t + R(\mu)x_t, \tag{17}$$

where $u_t(\theta) = u(t + \theta), \theta \in [-1, 0]$.
For $\psi \in C([0, 1], R^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0. \end{cases} \tag{18}$$

For $\phi \in C([-1, 0], R)$ and $\psi \in C([0, 1], R^*)$, define the bilinear form

$$< \psi, \phi >= \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \tag{19}$$

where $\eta(\theta) = \eta(\theta, 0)$. We have the following result on the relation between the operators $A = A(0)$ and A^*.

Lemma 3.1. *$A = A(0)$ and A^* are adjoint operators.*

The proof follows from (19). Here we omit it.
By the discussions in Section 2, we know that $\pm i\omega_0\tau_k$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0\tau_k$ and $-i\omega_0\tau_k$, respectively. By direct computation, we have the following result.

Lemma 3.2. *The vector $q(\theta) = e^{i\omega_0\tau_k\theta}$, $\theta \in [-1, 0]$, is the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_k$, and $q^*(s) = De^{i\omega_0\tau_k s}$, $s \in [0, 1]$, is the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0\tau_k$, moreover, $< q^*(s), q(\theta) >= 1$, where $D = \frac{1}{1-2\delta e^{i\omega_0\tau_k}}$.*

Next, we use the same notations as those in Hassard, Kazarinoff and Wan[22], and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of Eq.(1) when $\mu = 0$.

Define

$$z(t) =< q^*, x_t >, \; W(t, \theta) = x_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\} \tag{20}$$

on the center manifold C_0, and we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \tag{21}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \tag{22}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if x_t is real, we consider only real solutions. For solutions $x_t \in C_0$ of (1),

$$\begin{aligned} \dot{z}(t) &= <q^*(s), \dot{x}_t> = <q^*(s), A(0)u_t + R(0)u_t> \\ &= <q^*(s), A(0)x_t> + <q^*(s), R(0)x_t> \\ &= <A^*q^*(s), x_t> + \bar{q^*}(0)R(0)x_t \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{q^*}(\xi-\theta)d\eta(\theta)A(0)R(0)x_t(\xi)d\xi \\ &= <i\omega_0\tau_k q^*(s), x_t> + \bar{q^*}(0)f(0, x_t(\theta) \\ &\overset{\text{def}}{=} i\omega_0\tau_k z(t) + \bar{q^*}(0)f_0(z(t), \bar{z}(t)). \end{aligned} \tag{23}$$

That is

$$\dot{z}(t) = i\omega_0\tau_k z + g(z, \bar{z}), \tag{24}$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{25}$$

Hence, we have

$$g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) = \bar{q}^*(0)f_0(0, x_t) = \bar{D}f_0(0, x_t), \tag{26}$$

where

$$f_0(0, x_t) = \tau_k\left[-3\sqrt{\delta\epsilon}x_t^2(-1) - \epsilon x_t^3(-1)\right].$$

Noticing that $x_t(\theta) = W(t, \theta) + zq(\theta) + \bar{z}\bar{q}$ and $q(\theta) = e^{i\omega_0\tau_k\theta}$, we have

$$\begin{aligned} x_t(0) &= z + \bar{z} + W_{20}(0)\frac{z^2}{2} + W_{11}(0)z\bar{z} \\ &\quad + W_{02}(0)\frac{\bar{z}^2}{2} + \cdots, \\ x_t(-1) &= e^{-i\omega_0\tau_k}z + e^{i\omega_0\tau_k}\bar{z} + W_{20}(-1)\frac{z^2}{2} \\ &\quad + W_{11}(-1)z\bar{z} + W_{02}(-1)\frac{\bar{z}^2}{2} + \cdots. \end{aligned}$$

It follows from (26) that

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0)F_0(z, \bar{z}) = \bar{D}f_0(0, x_t) \\ &= -\bar{D}\tau_k\left[3\sqrt{\delta\epsilon}e^{-i\omega_0\tau_k}z^2 + 6\mathrm{Re}\{e^{i\omega_0\tau_k}\}z\bar{z}\right. \\ &\quad \left. + 3\sqrt{\delta\epsilon}e^{i\omega_0\tau_k}\bar{z}^2\right] - \bar{D}\tau_k\left\{3\sqrt{\delta\epsilon}\left[W_{11}(-1)\right.\right. \\ &\quad \left. + \frac{1}{2}W_{20}(-1) + W_{11}(0)e^{-i\omega_0\tau_k}\right] \\ &\quad \left. + \frac{1}{2}W_{20}(0)e^{i\omega_0\tau_k} + 3\epsilon e^{-i\omega_0\tau_k}\right\}z^2\bar{z} + \cdots. \end{aligned}$$

Then we obtain

$$\begin{aligned} g_{20} &= -6\bar{D}\tau_k\sqrt{\delta\epsilon}e^{-i\omega_0\tau_k}, g_{11} = -6\bar{D}\tau_k\mathrm{Re}\{e^{i\omega_0\tau_k}\}, \\ g_{02} &= 6\sqrt{\delta\epsilon}e^{i\omega_0\tau_k}, \\ g_{21} &= -2\bar{D}\tau_k\left\{3\sqrt{\delta\epsilon}\left[W_{11}(-1) + \frac{1}{2}W_{20}(-1)\right.\right. \\ &\quad \left.\left. + W_{11}(0)e^{-i\omega_0\tau_k} + \frac{1}{2}W_{20}(0)e^{i\omega_0\tau_k}\right] + 3\epsilon e^{-i\omega_0\tau_k}\right\}. \end{aligned}$$

For unknown $W_{11}(0), W_{11}(-1), W_{20}(0), W_{20}(-1)$ in g_{21}, we still need to compute them. From (17) and (20), we have

$$\begin{aligned} W' &= \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(\theta)\}, & -1 \le \theta < 0, \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(\theta)\} + f_0, & \theta = 0 \end{cases} \\ &\overset{\text{def}}{=} AW + H(z, \bar{z}, \theta), \end{aligned} \tag{27}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{28}$$

It follows from (27) and (28) that

$$(A - 2i\omega_0\tau_k)W_{20}(\theta) = -H_{20}(\theta), \tag{29}$$

$$AW_{11}(\theta) = -H_{11}(\theta). \tag{30}$$

We know that for $\theta \in [-1, 0)$,

$$\begin{aligned} H(z, \bar{z}, \theta) &= -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) \\ &= -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \end{aligned} \tag{31}$$

Comparing the coefficients of (31) with (28) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{32}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{33}$$

From (29),(32) and the definition of A , we get

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta). \tag{34}$$

Noting that $q(\theta) = q(0)e^{i\omega_0\tau_k\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_1e^{2i\omega_k\theta}, \tag{35}$$

where E_1 is a constant vector. Similarly, from (30), (33) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g_{11}}\bar{q}(\theta), \tag{36}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_2. \tag{37}$$

where E_2 is a constant vector.

In what follows, we shall seek appropriate E_1, E_2 in (35), (37), respectively. It follows from the definition of A, (29) and (30) that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0) \tag{38}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{39}$$

where $\eta(\theta) = \eta(0, \theta)$.
In view of (27) and (28), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + 2\tau_k\left(-3\sqrt{\delta\epsilon}e^{-i\omega_0\tau_k}\right) \tag{40}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + 2\tau_k\left(-3\mathrm{Re}\{e^{i\omega_0\tau_k}\}\right). \tag{41}$$

From (38), (39) and the definition of A, we have

$$\begin{cases} -2\delta W_{20}(-1) = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0), \\ -2\delta W_{11}(-1) = -H_{11}(0). \end{cases} \tag{42}$$

Noting that

$$\left(i\omega_0\tau_k I - \int_{-1}^{0} e^{i\omega_0\tau_k\theta} d\eta(\theta)\right) q(0) = 0, \tag{43}$$

$$\left(-i\omega_0\tau_k I - \int_{-1}^{0} e^{-i\omega_0\tau_k\theta} d\eta(\theta)\right) \bar{q}(0) = 0 \tag{44}$$

and substituting (35) and (40) into the first equation of (42), we have

$$\left(2i\omega_0\tau_k I - \int_{-1}^{0} e^{2i\omega_0\tau_k\theta} d\eta(\theta)\right) E_1 = -2\left(3\tau_k\sqrt{\delta\epsilon}e^{-i\omega_0\tau_k}\right). \tag{45}$$

That is

$$\left(2i\omega_0 + 2\delta e^{-2i\omega_0\tau_k}\right) E_1 = -2\left(3\sqrt{\delta\epsilon}e^{-i\omega_0\tau_k}\right).$$

Thus

$$E_1 = -\frac{3\sqrt{\delta\epsilon}e^{-i\omega_0\tau_k}}{i\omega_0 + \delta e^{-2i\omega_0\tau_k}}. \tag{46}$$

Similarly, substituting (37) and (41) into the second equation of (42), we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right) E_2 = -2\left(3\tau_k \mathrm{Re}\{e^{i\omega_0\tau_k}\}\right). \tag{47}$$

That is

$$2\delta E_2 = 2\left(3\mathrm{Re}\{e^{i\omega_0\tau_k}\}\right),$$

which leads to

$$E_2 = \frac{3\mathrm{Re}\{e^{i\omega_0\tau_k}\}}{\delta}. \tag{48}$$

In view of (35), (37), (46) and (48), we can calculate g_{21} and derive the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_k}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_k)\}}, \beta_2 = 2\mathrm{Re}(c_1(0)), \\ T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\tau_k)\}}{\omega_0\tau_k}, \end{aligned}$$

which give a description of the Hopf bifurcation periodic solutions of (1) at $\tau = \tau_k$ on the center manifold. From the discussion above, we have the following result.

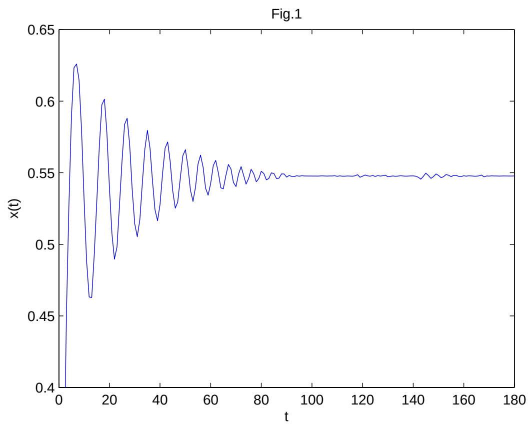
Theorem 3.3. *For system (1), if (H) holds, the periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$); The bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); The periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).*

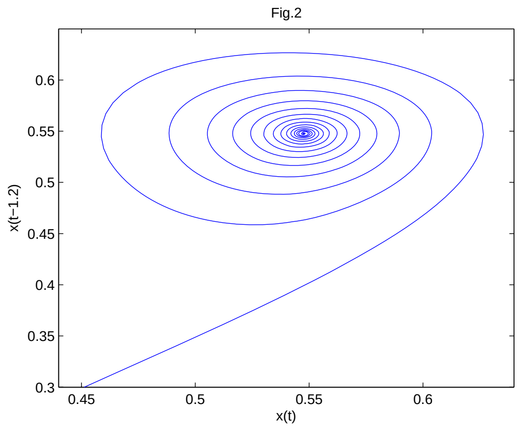
IV. Numerical Examples

In this section, to illustrate the analytical results found, let us consider the following special case of system (1)

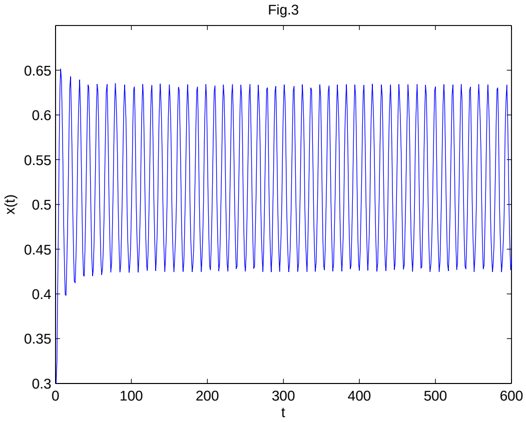
$$\frac{dx(t)}{dt} = 0.6x(t-\tau) - 2[x(t-\tau)]^3, \tag{49}$$

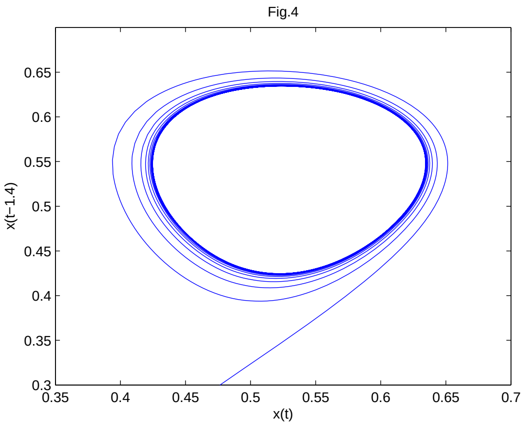
which has a unique positive equilibrium $x^* \approx 0.5477$. It follows from Theorem 2.1 that $\omega_0 \approx 1.2083, \tau_0 \approx 1.3002$. By means of Matlab 7.0, we get $\lambda'(\tau_0) \approx 1.2055 - 0.3847i, c_1(0) \approx -1.4535 - 7.0342i, \mu_2 \approx 1.2057, \beta_2 \approx -2.9070, T_2 \approx 0.5401$. Then it follows that $\mu_2 > 0$ and $\beta_2 < 0$. By Theorem 2.1, we know that the positive equilibrium is stable when $\tau < \tau_0$. Figs.1-2 show that the positive equilibrium $x^* \approx 0.5477$ is asymptotically stable when $\tau = 1.2 < \tau_0 \approx 1.3$. A Hopf bifurcation occurs when $\tau = \tau_0$, the positive equilibrium loses its stability and a periodic solution bifurcating from the positive equilibrium occurs for $\tau > \tau_0$. The bifurcation is supercritical and the bifurcating periodic solution is orbitally asymptotically stable. Figs.3-4 show that a family of periodic solutions bifurcate from the positive equilibrium $x^* \approx 0.5477$ when $\tau = 1.4 > \tau_0 \approx 1.3002$.





Figs.1-2 The trajectories graphs of system (49) with $\tau = 1.2 < \tau_0 \approx 1.3002$ and the initial value 0.8.





Figs.3-4 The trajectories graphs of system (49) with $\tau = 1.4 > \tau_0 \approx 1.3002$ and the initial value 0.8.

V. Conclusions

In this paper, we have investigated the properties of Hopf bifurcation in a nonlinear delay population model. By using the delay as bifurcation parameter, it has been shown that Hopf bifurcation occurs when the delay τ passes through some critical values $\tau = \tau_k, k = 0, 1, 2, \cdots$. This means that a class of periodic orbits bifurcates from the corresponding equilibrium. Moreover, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are derived by applying the normal form theory and center manifold theorem. To verify some of the mathematical results, we have taken an example for the model. Computer simulations are carried out for some artificial chosen data.

Acknowledgment

The author is grateful to supporting by National Natural Science Foundation of China (No.60902044), the Doctoral Foundation of Guizhou College of Finance and Economics(2010) and the Soft Science and technology Program of Guizhou Province(No.2011LKC2030).

References

[1] A. Uçar, *A prototype model for chaos studies*. Int. J. Eng. Sci. 40 (2002) 251-258.

[2] A. Uçar, *On the chaotic behavior of the prototype delayed dynamical system*. Chaos, Solitons and Fractals 16(2003) 187-194.

[3] M.S. Peng,*Bifurcation and chaotic behavior in the Euler method for a Uçar prototype delay model*. Chaos, Solitons and Fractals 22(2004) 483-493.

[4] C.G. LI, X.F. Liao and J.B. Yu. *Hopf bifurcation in a prototype delayed system*. Chaos, Solitons and Fractals 19 (2004) 779-787.

[5] X.P. Yan and W.T. Li. *Hopf bifurcation and global periodic solutions in a delayed predator-prey system*. Appl. Math. Comput. 177(1) (2006) 427-445.

[6] X.P. Yan and Y.D. Chu. *Stability and bifurcation analysis for a delayed Lotka-Volterra predator-prey system*. J. Comput. Appl. Math. 196(1) (2006) 198-210.

[7] X.P. Yan and C.H. Zhang. *Hopf bifurcation in a delayed Lokta-Volterra predator-prey system*. Nonlinear Anal.: Real World App. 9(1) (2008) 114-127.

[8] X.P. Yan and C.H. Zhang. *Direction of Hopf bifurcation in a delayed Lotka-Volterra competition diffusion system*. Nonlinear Anal.: Real World Appl. 10(5) (2009) 2758-2773.

[9] X.P. Yan and W.T. Li. *Bifurcation and global periodic solutions in a delayed facultative mutualism system*. Physica D 227(2007) 51-69.

[10] Y.L. Song and J.J. Wei. *Local Hopf bifurcation and global periodic solutions in a delayed predator-prey system*. J. Math. Anal. Appl. 301(1) (2005) 1-21.

[11] Y.L. Song, Y.H. Peng and J.J. Wei. *Bifurcations for a predator-prey system with two delays*. J. Math. Anal. Appl. 337(1) (2008) 466-479.

[12] Y.L. Song, S.L. Yuan and J.M. Zhang. *Bifurcation analysis in the delayed Leslie-Gower predator-prey system*. Appl. Math. Modelling 33(11) (2009) 4049-4061.

[13] Y.L. Song and S.L. Yuan. *Bifurcation analysis in a predator-prey system with time delay*. Nonlinear Anal.: Real World Appl.7(2) (2006) 265-284.

[14] S.L. Yuan and F.Q. Zhang. Stability and global Hopf bifurcation in a delayed predator-prey system. Nonlinear Anal.: Real World Appl. 11(2) (2010) 959-977.

[15] X.Y. Zhou, X.Y. Shi and X.Y. Song. *Analysis of non-autonomous predator-prey model with nonlinear diffusion and time delay*. Appl. Math. Comput. 196 (2008) 129-136.

[16] C.J. XU, X.H. Tang and M.X. Liao. *Stability and bifurcation analysis of a delayed predator-prey model of prey dispersal in two-patch environments*. Appl. Math. Comput. 216(10) (2010) 2920-2936.

[17] C.J. XU, X.H. Tang, M.X. Liao. and X.F. He. *Bifurcation analysis in a delayed Lokta-Volterra predator-prey model with two delays*. Nonlinear Dynam., doi: 10.1007/s11071-010-9919-8

[18] C.J. XU, M.X. Liao. and X.F. He. *Stability and Hopf bifurcation analysis for a Lokta-Volterra predator-prey model with two delays*. Int. J. Appl. Math. Comput. Sci. 21(2011) 97-107.

[19] C.J., Xu and Y.F. Shao, *Bifurcations in a predator-prey model with discrete and distributed time delay*. Nonlinear Dynam., doi: 10.1007/s11071-011-0140-1

[20] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*. INC: Academic Press, 1993.

[21] J. Hale, *Theory of Functional Differential Equations*. Springer-Verlag, 1977.

[22] B. Hassard, D. Kazatina and Y. Wan, *Theory and applications of Hopf bifurcation*. Cambridge: Cambridge University Press, 1981.

Changjin Xu is an associate professor of Guizhou College of Finance and Economics. He received his M. S. from Kunming University of Science and Technology, Kunming, in 2004 and Ph. D. from Central South University, Changsha, in 2010. His current research interests focus on the stability and bifurcation theory of delayed differential equation and periodicity of the functional differential equations and difference equations.