Tensorial Transformations of Double Gai sequence spaces

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Abstract—The precise form of tensorial transformations acting on a given collection of infinite matrices into another; for such classical ideas connected with the summability field of double gai sequence spaces. In this paper the results are impose conditions on the tensor $g$ so that it becomes a tensorial transformations from the metric space $\chi^2$ to the metric space $C$

Keywords—tensorial transformations, double gai sequences, double analytic, dual.

I. INTRODUCTION

Let $(x_{mn})$ be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^\infty x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^\infty x_{mn}$ is said to be convergent if and only if the double sequence $(s_{mn})$ is convergent, where

$$s_{mn} = \sum_{i,j=1}^m x_{ij} (m, n = 1, 2, 3, \ldots)$$

see[1]. We denote $w^2$ as the class of all complex double sequences $(x_{mn})$. Let $\Omega$ be the field of infinite matrices endowed with usual operations of pointwise addition and scalar multiplication.

A sequence $x = (x_{mn}) \in \Omega$ is said to be double analytic if

$$\sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$  

The vector space of all prime sense double analytic sequences are usually denoted by $\Lambda^2$. A sequence $x = (x_{mn}) \in \Omega$ is called a double gai sequence if

$$((m+n)! x_{mn})^{1/m+n} \rightarrow 0 \text{ as } m+n \rightarrow \infty.$$ 

We denote $\chi^2$ as the class of prime sense double gai sequences. The spaces $\Lambda^2$ and $\chi^2$ are metric spaces with metrics

$$d(x, y) = \sup_{mn} \{(|x_{mn} - y_{mn}|^{1/m+n}) : mn = 1, 2, \ldots\}$$

for all $x = (x_{mn})$ and $y = (y_{mn})$ in $\Lambda^2$ and

$$\hat{d}(x, y) = \sup_{mn} \{(m+n)! |x_{mn} - y_{mn}|^{1/m+n} : mn = 1, 2, \ldots\}$$

for all $x = (x_{mn})$ and $y = (y_{mn})$ in $\chi^2$, respectively.

$$\ell^2 = \{ x = (x_{mn}) \in \Omega : \sum_{mn} |x_{mn}| < \infty \}.$$ 

The space $\chi^2$ can be then regarded as the space of gai functions of two variables equipped with the topology of uniform convergence on compact sets $C \times C$, where $C$ is the complex plane. These spaces are known to be Frechet spaces.

For any double sequence $x = (x_{mn})$ the $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by

$$x^{[m,n]} = \sum_{i,j=m}^{m+n} x_{ij} \zeta_{ij}$$

for all $m, n \in \mathbb{N}$, where

$$\zeta_{mn} = \begin{pmatrix} 0, 0, \ldots, 0, 0, \ldots \cr 0, 0, \ldots, 0, 0, \ldots \cr \vdots \cr 0, 0, \ldots, 1, -1, 0, \ldots \cr 0, 0, \ldots, 0, 0, \ldots \end{pmatrix}$$

with 1 in the $(m, n)^{th}$ and -1 $(m+1, n+1)^{th}$ position and zero other wise.

An infinite matrix shall be denoted by $x = (x_{mn})$

$$\begin{pmatrix} x_{00}, x_{01}, \ldots, x_{0n}, \ldots \cr x_{10}, x_{11}, \ldots, x_{1n}, \ldots \cr \vdots \cr x_{mn}, x_{m1}, \ldots, x_{mn}, \ldots \cr \vdots \cr \end{pmatrix}$$

where $x_{mn}$'s belong to the field $K$ of scalars. Denote by $N$ the set of all non-negative integers. Thus $\Omega$ is a vector space over $K$. By a matrix space $X$ we mean any subspace $\Omega$. The matrix space generated by $\{\zeta_{mn} : m, n \in N\}$ shall be denoted by $\varphi$. If $N \in N$ and $x \in \Omega$, we define

$$x^N = \sum_{0\leq m+n<N} x_{mn} \zeta_{mn}$$

and call it as the $N^{th}$-place section of the matrix $x$. For a matrix space $X$, we define $X'$ by

$$X' = \{ y = (y_{mn}) : y \in \Omega \text{ with } \sum_{mn} |x_{mn} y_{mn}| < \infty \text{ for all } x \in X \}$$

where

$$\lim_{N \to \infty} \sum_{0\leq m+n<N} x_{mn} y_{mn} = \sum_{0\leq m+n<N} x_{mn} y_{mn}$$

and term it as the $K$-dual of $X$. Clearly $X'$ is a vector space over $K$ and contains $\varphi$. We assume that each matrix space $X$ contains $\varphi$ under
this assumption, X and X′ form a dual system which express as \((X, X')\). Hence, the weak topology \(\sigma (X, X')\), the Mackey topology \(\tau (X, X')\), the strong topology \(\beta (X, X')\) and so on.

**K- normal and K- perfect matrix spaces:**

A matrix space is called K- normal provided it tends to infinity with \(m, n\) is well defined for all \(m, n\). Indeed, if \(X\) is a tensor of order four, which relate various matrix spaces. We assume that its multiplication with any preassigned matrix as \(X\) for \(y\) tends to infinity. For \(x\), define the matrix \(X\) by \(X^{\prime} = X\); observe that \(X \subset X^{\prime}\) is always true.

**II. PRELIMINARIES**

Some initial works on double sequence spaces is found in Bromwich[3]. Later on, it was investigated by Hardy[5], Moricz[7], Moricz and Radoheas[8], Basarir and Solank[2], Tripathy[10], Colak and Turkmenoglu[4], Turkmenoglu[11], Patterson [9] and many others. In this paper we study some of the properties of transformations resulting from a tensor of order four, which relation various matrix spaces.

**III. MAIN RESULTS**

**A. Theorem**

We have \((\chi_2)^{\prime} = \Lambda_2\) and \((\Lambda_2)^{\prime} = \chi_2\). Thus \(\chi_2\) and \(\Lambda_2\) are K- perfect

**Proof:** We prove only \((\chi_2)^{\prime} = \Lambda_2\); the proof of \((\Lambda_2)^{\prime} = \chi_2\) is similar. Now observe that \(\Lambda_2 \subset \chi_2^{\prime}\) is obvious. For \((\chi_2)^{\prime} \subset \Lambda_2\), let \(x \in (\chi_2)^{\prime}\) and \(x \notin \Lambda_2\). For each integer \(i \geq 1\) there exist sequences \((m_i)\) and \((n_i)\) (atleast one of which tends to infinity with \(i\)) such that

\[
|x_{m_i, n_i}| > \frac{i^{p+q}}{(m_i, n_i)}
\]

Define the matrix \(y\) by

\[
y_{mn} = \left\{ \begin{array}{ll}
\frac{i^{-m_i-n_i}}{i^{p+q}}, & \text{if } m = m_i, n = n_i; \\
0, & \text{otherwise}
\end{array} \right.
\]

Thus \(y \in \chi_2\). However \(\sum \sum |x_{m, n}| = \infty\) and \(x \notin (\chi_2)^{\prime}\), a contradiction. This completes the proof.

**B. Theorem**

Suppose eqn. (1) is true for each \(y \in \chi_2\). Then \(x = (x_{mn}) \in C\) if and only if there exists a constant \(M > 0\) such that

\[
|\chi_2^{00}_{mn}| \leq M, \text{ for all } m, n, p, q \in \mathbb{N},
\]

and

\[
\lim_{m+n \to \infty} (\chi_2^{pq})_{mn} = \Lambda_2^{pq} \exists \text{ for every } p, q \geq 0
\]

**Proof:** The proof of the sufficiency part is straightforward and is therefore omitted.

For converse, let \(x \in C\) where \(x = (x_{mn})\) is given by eqn.(1). For \(y \in \chi_2\), define the matrix \(f = (f_{mn})\) of functionals by

\[
f_{mx} (y) = x_{mn} = \sum_{p+q \geq 0} (\chi_2^{pq})_{mn} y_{pq}.
\]

Since the set

\[
\left\{ |\chi_2^{00}_{mn}|, ((p+q) |\chi_2^{pq}_{mn}|)^{1/p+q}, p+q \geq 1 \right\}
\]

is analytic for fixed pair of integers \(m, n\); it follows that the functionals \(f_{mx}\) are continuous. Moreover, therese functionals are pointwise analytic. Therefore by uniform boundness principle there exists a ball \(B_2(z)\) such that for all \(y \in B_2(z)\).

\[
|f_{mx} (y)| \leq M, \text{ for all } m, n \geq 0
\]

where \(M\) is a constant and all \(y\) with \(|y| \leq \epsilon\). Choosing \(y\) to be the matrices \(y^{pq}\) for \(p+q \geq 0\) respectively, where \(y^{pq} = (\epsilon_{ij})\)

\[
\epsilon_{ij} = \left\{ \begin{array}{ll}
\frac{e^{i+q}}{(p+q)!}, & \text{if } i = p, j = q; \\
0, & \text{otherwise}
\end{array} \right.
\]

when \(p+q > 0\) and \(y^{00} = (\epsilon_{ij})\), \(\epsilon_{ij} = \epsilon, \epsilon_{ij} = 0, i + j \geq 1\). We obtain \(|\chi_2^{00}_{mn}| \leq M\) for all \(m, n \geq 0\) and \(|\chi_2^{pq}_{mn}|^{1/(p+q)} \leq M\), for all \(m, n \geq 0\) and \(p+q \geq 1\). Thus

\[
|\chi_2^{00}_{mn}| \leq M, (p+q)! |\chi_2^{pq}_{mn}|^{1/p+q} \leq M^{1/p+q} \times \frac{1}{(p+q)!}
\]

for \(m+n \geq 0\) and \(p+q > 0\). Since

\[
M^{1/p+q} \times \frac{1}{(p+q)!} \leq M \quad \text{for } p+q > 0
\]

it follows that

\[
|\chi_2^{00}_{mn}|, (|\chi_2^{pq}_{mn}|)^{1/p+q} \leq \frac{1}{(p+q)!} \times M^{1/p+q} \text{ for } m+n \geq 0
\]

This proves eqn. (2). The condition of eqn. (3) obviously follows.

This completes the proof.

**C. Theorem**

Let eqn.(1) be true for \(y \in \ell^2\). Then \(x = (x_{mn}) \in \chi_2\) if and only if

\[
(m+n)! |\chi_2^{pq}_{mn}|^{1/m+n} \to 0 \text{ as } m+n \to \infty
\]

uniformly in \(p\) and \(q\).

**Proof:** Sufficiency follows by straightforward calculations. For necessity, assume that eqn. (4) is not true. Then for \(\epsilon > 0\), and any \(N \in \mathbb{N}\), there exist integers \(m, n\) and \(p, q\) such that \(m+n > N\) and

\[
(m+n)! |\chi_2^{pq}_{mn}|^{1/m+n} > \epsilon
\]

Since a maps \(\ell^2\) in \(\chi_2\), it follows a transforms \(\ell^2\) into itself and therefore
Then we write
\[ w_{mn} = \sup_{p,q \geq 0} |x^2|_{mn}^{pq} : p + q \geq 0 \] 
\[ \sup \left\{ \sum_{m+n \geq 0} \chi_{mn}^{pq} : p + q \geq 0 \right\} \leq M. \]

We also have
\[ \left( (m + n)! \right)^{1/m + n} \rightarrow 0 \text{ as } m + n \rightarrow \infty \] 
(7)

for each fixed \( p \) and \( q \). By eqn. (5) we can find \( m_{n1} \) and \( p_{1q} \) such that
\[ \left( m_{n1} + 1 \right)! \chi_{mn1}^{pq} \left( \chi_{mn1}^{pq} \right)^{1/m_{n1} + 1} > \epsilon / 2 \] 
(8)

Now from the relations eqn. (5) to eqn. (7), choose \( m_{2n1} \) sufficiently large with \( m_{2n1} + 2 > m_{1n1} + n_{2} + p_{2} + q_{2} > p_{1} + q_{1} \) such that
\[ \left( m_{2n1} + 2 \right)! \chi_{mn2}^{pq} \left( \chi_{mn2}^{pq} \right)^{1/m_{2n1} + 2} > \epsilon / 2 \] 
(9)

and
\[ \left( m_{2n1} + 2 \right)! \chi_{mn2}^{pq} \left( \chi_{mn2}^{pq} \right)^{1/m_{2n1} + 2} < \frac{\epsilon}{16} \] 
(10)

Proceeding in this way, we get sequences \( \{m_{k}\} \), \( \{p_{k}\} \) and \( \{q_{k}\} \) with \( m_{k} + n_{k} > m_{k-1} + n_{k-1}, p_{k} + q_{k} > p_{k-1} + q_{k-1} \) such that
\[ \left( m_{k} + n_{k} \right)! \chi_{mnk}^{pq} \left( \chi_{mnk}^{pq} \right)^{1/m_{k} + n_{k}} > \epsilon / 8 \] 
(11)

Let us now introduce the matrix \( y = (y_{pq}) \in \ell^2 \) as follows
\[ y_{pq} = \begin{cases} \frac{1}{m_{k} + n_{k}}, & \text{if } p = p_{k}, q = q_{k} \text{, } k = 1, 2, 3, \ldots \; \text{or} \\ 0, & \text{otherwise} \end{cases} \]

It is easily verified that \( x = (x_{mn}) \notin \chi^2 \) where
\[ x_{mn} = \sum_{p+q \geq 0} \chi_{mn}^{pq} y_{pq} \text{ for all } m,n \geq 0 \]

Indeed,
\[ \left( m_{k} + n_{k} \right)! \chi_{mnk}^{pq} \left( \chi_{mnk}^{pq} \right)^{1/m_{k} + n_{k}} \]
\[ \geq \frac{1}{2} \left( m_{k} + n_{k} \right)! \chi_{mnk}^{pq} \left( \chi_{mnk}^{pq} \right)^{1/m_{k} + n_{k}} - \]
\[ \left( m_{k} + n_{k} \right)! \sum_{j<k} \chi_{mnk}^{pq} y_{pq} \chi_{mnk}^{pq} \left( \chi_{mnk}^{pq} \right)^{1/m_{k} + n_{k}} - \]
\[ \left( m_{k} + n_{k} \right)! \sum_{j>l} \chi_{mnk}^{pq} y_{pq} \chi_{mnk}^{pq} \left( \chi_{mnk}^{pq} \right)^{1/m_{k} + n_{k}} \]
\[ > \frac{\epsilon}{8} - \frac{\left( k-1 \right)!}{8k^2} - \frac{\epsilon}{8} = \frac{\epsilon}{8} \]

for all \( k \geq 1 \). Hence it is a contradiction and the result follows.

Similarly, we can prove the following result

D. Theorem

Let eqn.(1) be true for \( y \in \ell^2 \). Then \( x = (x_{mn}) \in \Lambda^2 \) if and only if
\[ \left( (m + n)! \right)^{1/m + n} \leq M, \]
uniformly in \( p, q \) and \( m, n \); where \( M \) is a positive constant.

IV. CONCLUSION

Tensorial transformation of classical ideas connected with the field of double gai sequence spaces.

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REFERENCES

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