Lateral and Longitudinal Vibration of a Rotating Flexible Beam Coupled with Torsional Vibration of a Flexible Shaft

Khaled Alnefaie

Abstract—In this study, rotating flexible shaft-disk system having flexible beams is considered as a dynamic system. After neglecting nonlinear terms, torsional vibration of the shaft-disk system and lateral and longitudinal vibration of the flexible beam are still coupled through the motor speed. The system has three natural frequencies; the flexible shaft-disk system torsional natural frequency, the flexible beam lateral and longitudinal natural frequencies. Eigenvalue calculations show that while the shaft speed changes, torsional natural frequency of the shaft-disk system and the beam longitudinal natural frequency are not changing but the beam lateral natural frequency changes. Beam lateral natural frequency stays the same as the nonrotating beam lateral natural frequency \( \omega_l \) until the motor speed \( \omega_m \) is equal to \( \omega_p \). After then \( \omega_m \) increases and remains equal to the motor speed \( \omega_m \) until the motor speed is equal to the shaft-disk system natural frequency \( \omega_p \). Then the beam lateral natural frequency \( \omega_l \) becomes equal to the natural frequency \( \omega_p \) and stays same while the motor speed \( \omega_m \) is increased. Modal amplitudes and phase angles of the vibrations are also plotted against the motor speed \( \omega_m \).

Keywords—Rotor dynamics, beam-shaft coupling, beam vibration, flexible shaft.

I. INTRODUCTION

VIBRATION problem of rotating beams has been a subject of extensive research due to a number of very important applications such as helicopter blades, turbine blades, and appendages of spinning satellites. References [7], [8], [10] have studied a model consisting of servomotor, harmonic drive, flexible shaft and a rigid manipulator arm. Transfer function of the system relating desired input rotation to the manipulator arm rotation is developed. Flexible system natural frequency and damping ratio together with PID control parameters appear in the transfer function. The possibility of a precise trajectory tracking is discussed and frequency response characteristics of the system with respect to some parameters are studied. Reference [5] have also modeled rotating Euler-Bernoulli type beam and studied residual vibration spectrum of the beam. It is shown that at certain frequencies of the rise function, residual vibration can be eliminated. For the same model shear force at the root of the beam is used as a feedback for the control system and parametric analysis is done, the effect of shear force feedback control strategy on the beam tip vibration is studied. References [12], [13] have studied coupling effect of a flexible link and a flexible joint in one-link rotating structure. Two nondimensional parameters are defined; the ratio of a bending stiffness of the link to the torsional stiffness of the rotor and the ratio of moment of inertia of the link to the rotor. Unconstrained and constrained modal expansions are compared. Reference [2] has developed general model to describe the rotating blade vibration under the effect of shaft torsional vibration. Reference [1] have studied mathematical model for a flexible arm undergoing large planar flexural deformation, continuously rotating under the effect of a hub torque and supported by a flexible base. Reference [3] have developed a new approach based on a linear quadratic estimator technique for estimating the vibration of any point on the span of a rotating flexible beam mounted on a compliant hub in the presence of process and measurement noise. Reference [11] have studied coupled nonlinear equations of motion of a coupled elastic shaft-elastic beam model in a very general fashion considering the influence of rotor, shaft, hub, beam and payload as well as geometric stiffness terms which arise from both centrifugal and Coriolis accelerations. Their solution concentrates on the effect of two parameters representing the mass and stiffness ratios of the manipulator system on its driveline. References [4], [6], [9] analyzed a servomotor driven coupled elastic shaft–elastic beam system. The model consists of a servomotor, disk, and an elastic beam attached to the disk and separated by an elastic shaft. Equations of motion are derived with respect to the generalized coordinates of the elastic shaft, elastic beam and the servomotor rotation. Nonlinear terms coming from Coriolis, normal and tangential accelerations are retained in the equations. Change of eigenvalues of the system with respect to the control parameters and also beam tip vibrations are investigated.

In this study, rotating flexible shaft-disk system having flexible beams is considered as a dynamic system. Equations of the dynamic system are derived considering that the shaft is driven with a motor. After neglecting nonlinear terms, torsional vibration of the shaft-disk system and lateral and longitudinal vibration of the flexible beam are still coupled through the motor speed. The change of natural frequencies and modal amplitudes of the system are investigated with respect to the motor speed.

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II. FORMULATION

Fig. 1 shows the dynamic system consists of a disk having flexible beams circumferentially connected to it and the disk is connected a motor through flexible shaft. \( J_m \) is the motor inertia, \( J_s \) is the disk inertia, \( l_b \) is the beam length, \( l_d \) is the shaft length, \( \theta_m \), \( \theta_s \) and \( \theta_d \) are the motor, flexible shaft and disk rotation, respectively. Front view of the system shows the configuration of the disk and the flexible beam. OXY is the fixed frame of reference, Oxyz is the rotating frame attached to the disk at disk center. \( r_d \) is the radius of the disk, \( r_d + x + u(x,t) \) and \( y(x,t) \) are the coordinates of the flexible beam unit mass \( m \) with respect to the rotating coordinates Oxyz. \( u(x,t) \) and \( y(x,t) \) are the longitudinal and lateral displacements of unit mass \( m \).

\[ \begin{align*}
\text{Fig. 1 Model of elastic shaft-disk-elastic beam system}
\end{align*} \]

The kinetic energy of the system can be written as

\[ T = \frac{1}{2} J_m \dot{\theta}_m^2 + \frac{1}{2} J_s \dot{\theta}_s^2 + N \int_{r_d}^{l_b} \left( \frac{1}{2} m v^2(x,t) \right) dx 
+ \frac{1}{2} \rho I_s \dot{\phi}^2(z,t) dz \]  
(1)

In (1) \( N \) is the number of flexible beams attached to the disk, \( v \) is the velocity of the unit mass \( m \), \( \rho \) is the density of beam material, \( I_s \) is the polar moment of inertia of the flexible shaft. The potential energy of the system can be written as

\[ U = \frac{1}{2} N \int_{r_d}^{l_b} EI \dot{\phi}^2(x,t) dx 
+ \frac{1}{2} N \int_{r_d}^{l_b} E Al^2 \dot{\phi}^2(z,t) dz \]  
(2)

In (2) \( EI \) is the rigidity of the elastic beam, \( GI_s \) is the rigidity of the elastic shaft, \( E A \) is the longitudinal rigidity of the beam, \( \psi(z,t) \) is the elastic rotation of the shaft. The disk and the shaft rotation can be written as the sum of the motor rotation and the flexible shaft rotation as

\[ \begin{align*}
\theta(z,t) &= \theta_m(t) + \psi(z,t) \\
\theta_s(t) &= \theta_s(t) + \psi(l_s, t)
\end{align*} \]  
(3)

It is assumed that the elastic motion of the beam and the shaft are sum of the orthogonal modes

\[ \begin{align*}
\psi(z,t) &= \sum_{i=1}^{n} \phi_i(z) q_i(t) \\
y(x,t) &= \sum_{i=1}^{n} \phi_i(x) q_i(t) \\
u(x,t) &= \sum_{i=1}^{n} \phi_i(u_i) q_i(t)
\end{align*} \]  
(4)

Here \( \phi_i(z) \), \( \phi_i(x) \) and \( \phi_i(u_i) \) are orthogonal modes, \( q_i(t) \), \( q_i(t) \) and \( q_i(t) \) are general coordinates of the elastic shaft and elastic beam, respectively. In the following derivations, only first mode is considered.

Position vector of the beam unit mass \( m \) can be written as

\[ \vec{r} = (r_d + x + u) \hat{i} + y \hat{j} \]  
(5)

The velocity of the unit mass \( m \) can be found by taking time derivative of the vector \( \vec{r} \) which is

\[ \vec{v} = \dot{\vec{r}} = (\dot{u} + y \dot{\theta}_d) \hat{i} + \dot{y} \hat{j} \]  
(6)

If (3), (4) and (6) are used the kinetic and the potential energies of (1) and (2) will be

\[ \begin{align*}
T &= \frac{1}{2} J_m \dot{\theta}_m^2 + \frac{1}{2} J_s \dot{\theta}_s^2 + \frac{1}{2} \rho I_s \dot{\phi}_1^2 \] 
+ \frac{1}{2} \int_{r_d}^{l_b} \left( m v^2(x,t) + \dot{\phi}_1^2(z,t) \right) dx 
+ \frac{1}{2} \rho I_s \dot{\phi}_1^2 \]  
(7)

\[ U &= \frac{1}{2} N \int_{r_d}^{l_b} EI \dot{\phi}_1^2 dx 
+ \frac{1}{2} N \int_{r_d}^{l_b} GI_s \dot{\phi}_1^2 dx \]  
(8)

In (7) \( \dot{\phi}_1 = \dot{\phi}(l_s) \). Lagrange equations are used to obtain dynamic equations of the system with respect to the generalized coordinates. Viscous damping is introduced through \( \zeta_s \), \( \zeta_r \) and \( \zeta \) which are lateral, longitudinal and torsional motion damping ratios, respectively. After ignoring nonlinear terms, the following equations are obtained

\[ \ddot{\tilde{q}}_s + 2 \zeta_s \omega_s \dot{\tilde{q}}_s + (\alpha_s^2 + \omega_s^2) \tilde{q}_s = 0 \]  
(9)

\[ \ddot{\tilde{q}}_b + 2 \zeta_r \omega_r \dot{\tilde{q}}_b + (\alpha_r^2 + \omega_r^2) \tilde{q}_b + 2 \alpha_r \omega_r \dot{\tilde{q}}_b = 0 \]  
(10)
\[ \left( J_{y}/J_{r} \right) \alpha_{y} + \left( NJ_{10}/J_{r} \right) \alpha_{y} + 1 \dddot{\phi}_{y} + 2 \dddot{\phi}_{y} \omega_{y} + \dddot{\phi}_{y} + \right) \left( 2NJ_{y}/J_{r} \right) \lambda \alpha_{y} \phi_{y} + \left( 2NJ_{y}/J_{r} \right) \lambda + (1/2) \alpha_{y} \omega_{y} \frac{d\phi_{y}}{dt} = 0 \] (11)

In these equations \( \omega_{y} \) is the motor speed, \( \omega_{b} \) and \( \omega_{n} \) are the first natural frequencies of lateral and longitudinal vibrations of the beam and \( \omega_{a} \) is the torsional vibration natural frequency of the elastic shaft. The following are the definitions of other parameters:

\[ \bar{q}_{y} = q_{y}/l_{b}, \quad \bar{q}_{a} = q_{a}/l_{b}, \quad \lambda = r_{y}/l_{b}, \]

\[ \alpha_{1} = \int_{0}^{l_{b}} \phi_{y} d\zeta \int_{0}^{l_{b}} \phi_{y} d\zeta, \quad \alpha_{2} = \int_{0}^{l_{b}} \phi_{y} d\zeta \int_{0}^{l_{b}} \phi_{y} d\zeta, \]

\[ \alpha_{3} = \int_{0}^{l_{b}} \phi_{y} d\zeta \int_{0}^{l_{b}} \phi_{y} d\zeta, \quad \alpha_{4} = \int_{0}^{l_{b}} \phi_{y} d\zeta \int_{0}^{l_{b}} \phi_{y} d\zeta, \]

\[ \alpha_{5} = \ddot{\phi}_{b} \int_{0}^{l_{b}} \phi_{y} d\zeta, \]

\[ \alpha_{6} = \int_{0}^{l_{b}} \phi_{y} d\zeta \int_{0}^{l_{b}} \phi_{y} d\zeta, \quad \alpha_{7} = \int_{0}^{l_{b}} \phi_{y} d\zeta \int_{0}^{l_{b}} \phi_{y} d\zeta \] (12)

Then the characteristic equation of the torsional vibration of the beam will be

\[ (\omega/c) \cos(\omega_{y}/c) = 0, \quad c = \sqrt{E/\rho} \] (19)

Concerning boundary conditions of the elastic shaft at the disk end, restoring moment is equal to the total inertial moment of the shaft, disk and beams.

Characteristic equation of the torsional vibration of elastic shaft will be

\[ \beta_{y} \tan \beta_{y} = J_{y}/J_{r} \] (23)

then torsional natural frequency of the shaft is

\[ \omega_{y} = \beta_{y} \sqrt{G/\rho_{s}^{2}} \] (24)

III. SIMULATIONS

Parameters selected for the simulations are listed in Table I.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_{y} )</td>
<td>disk radius</td>
<td>0.5 m</td>
</tr>
<tr>
<td>( m_{d} )</td>
<td>disk mass</td>
<td>120 kg</td>
</tr>
<tr>
<td>( l_{b} )</td>
<td>beam length</td>
<td>1.0 m</td>
</tr>
<tr>
<td>( W_{b} \times h_{b} )</td>
<td>beam width x height</td>
<td>0.020 x 0.005 m</td>
</tr>
<tr>
<td>( l_{s} )</td>
<td>shaft length</td>
<td>0.1, 0.5, 1.0 m</td>
</tr>
<tr>
<td>( d_{s} )</td>
<td>shaft diameter</td>
<td>0.06 m</td>
</tr>
<tr>
<td>( E )</td>
<td>modulus of elasticity</td>
<td>207 GPa</td>
</tr>
<tr>
<td>( G )</td>
<td>modulus of shear</td>
<td>79 GPa</td>
</tr>
</tbody>
</table>
Three shaft lengths are used for the simulation: 0.1 m, 0.5 m, and 1.0 m. Beam lateral and longitudinal natural frequencies, shaft torsional natural frequency and shaft-disk system torsional natural frequency with respect to the three different shaft lengths are tabulated in Table II.

### Table II

<table>
<thead>
<tr>
<th>$l$ (m)</th>
<th>$K$</th>
<th>$\omega_b$ (rad/s)</th>
<th>$\omega_m$ (rad/s)</th>
<th>$\omega_T$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>23308</td>
<td>26.3</td>
<td>8144.4</td>
<td>50314</td>
</tr>
<tr>
<td>0.5</td>
<td>4662</td>
<td>26.3</td>
<td>8144.4</td>
<td>10063</td>
</tr>
<tr>
<td>1.0</td>
<td>2331</td>
<td>26.3</td>
<td>8144.4</td>
<td>50314</td>
</tr>
</tbody>
</table>

Equations of the system given in (9), (10) and (11) can be put in a matrix form

$$[M]\ddot{q} + [C]q + [K]q = \{F\}$$

In open form, equations are

$$\begin{bmatrix}
1 & 0 & M_{13} & \bar{q}_s \\
0 & 1 & 0 & \bar{q}_s \\
M_{31} & 0 & M_{33} & \bar{q}_s \\
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_s \\
\ddot{q}_s \\
\ddot{q}_s \\
\end{bmatrix}
+ 2\zeta_b\omega_b
\begin{bmatrix}
C_{12} \\
C_{21} \\
C_{31} \\
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
\end{bmatrix}
+ 2\zeta_b\omega_b
\begin{bmatrix}
K_{12} \\
K_{21} \\
K_{31} \\
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
\end{bmatrix}
= \{f_2\}$$

Terms $M_{13}$, $M_{31}$, $M_{33}$, $C_{12}$, $C_{21}$, $C_{23}$, $C_{31}$ and $f_2$ can be found in (9), (10) and (11). There is a dynamic coupling through mass matrix elements $M_{13}$, $M_{31}$, but they are constant. There is no static coupling through stiffness matrix $[K]$ but there are coupling terms in the damping matrix $[C]$ through $C_{23}$, $C_{31}$, $C_{12}$ and $C_{32}$ and they are functions of motor speed $\omega_m$. The eigenvalues of (25) can be found by constructing the system matrix, which is

$$[A] = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

Solution of the system matrix will produce three pairs of complex eigenvalues, which are

$$s_{1,2} = -\zeta_b\omega_b \pm \omega_b \sqrt{1 - \zeta_b^2} j$$

$$s_{3,4} = -\zeta_b\omega_b \pm \omega_b \sqrt{1 - \zeta_b^2} j$$

$$s_{5,6} = -\zeta_b\omega_b \pm \omega_b \sqrt{1 - \zeta_b^2} j$$

Since the roots are functions of the motor speed $\omega_m$ for each given $\omega_m$, $\zeta_b$, $\omega_b$, $\zeta_3$, $\omega_r$ and $\zeta_T$ can be calculated.

Preliminary calculations show that the roots are sensitive to shaft length that is why the calculations are carried out for three different shaft lengths of 0.1m, 0.5m and 1.0m. Table II lists the rigidity factor $K$, the beam lateral natural frequency $\omega_b$, the beam longitudinal natural frequency $\omega_m$, the shaft torsional natural frequency $\omega_r$ and the torsional natural frequency of the shaft-disk system $\omega_T$. Since the rigidity factor is very high for all shaft lengths, beam lateral natural frequency is not affected with the selected shaft lengths.

Fig. 2 shows how the lateral natural frequency $\omega_b$ changes with the motor speed. When the motor speed is zero, nonrotating beam lateral natural frequency $\omega_b$ is 26.3 rad/s for all shaft lengths. $\omega_b$ stays the same as the nonrotating beam lateral natural frequency until the motor speed $\omega_m$ is equal to $\omega_b$. Then, $\omega_b$ starts increasing but remains equal to the motor speed $\omega_m$ until the motor speed is equal to the shaft-disk system natural frequency $\omega_T$. After then the beam lateral natural frequency $\omega_b$ becomes equal to the shaft-disk system natural frequency $\omega_T$ and stays same while the motor speed $\omega_m$ is increased.

Fig. 3 shows how beam lateral vibration damping ratio $\zeta_b$ is changing with the motor speed. Nonrotational damping ratio of 0.02 is assumed. While the motor speed is increased damping ratio decreases, when the motor speed reaches to shaft-disk system natural frequency $\omega_T$ increases back to its original value. For shorter shaft lengths, damping ratio is lowered more.
Fig. 3 The beam lateral vibration damping ratio with respect to motor speed

Fig. 4 shows the settling time of the beam lateral vibration. The settling time is defined as

\[ t_s = \frac{4}{\zeta_b \omega_b} \]  

(29)

Fig. 4 The settling time of the beam lateral vibration with respect to motor speed

Since the damping ratio \( \zeta_b \) and the natural frequency \( \omega_b \) are changing with the motor speed, settling time of the beam lateral vibrations will also change. Settling time plot shows that beam vibration settling time is much shorter when the motor speed reaches the natural frequency of the shaft-disk system \( \omega_T \).

Generalized coordinates \( q_b \), \( q_u \) and \( q_s \) can be written in terms of eigenvalues and eigenvectors as

\[ \{q\} = [E] \{S\} \]  

(30)

Here \( \{q\} = \{q_b, q_u, q_s\}^T \), \( [E] \) is 3x6 eigenvector matrix and \( \{S\} \) is 6x6 diagonal eigenvalue matrix. For example \( q_b \) in open form is

\[ q_b = E_{b1} e^{s_{b1} t} + E_{b2} e^{s_{b2} t} + E_{b3} e^{s_{b3} t} + E_{b4} e^{s_{b4} t} + E_{b5} e^{s_{b5} t} + E_{b6} e^{s_{b6} t} \]  

(31)

Here \( E_{b1} \) and \( E_{b2} \) are complex conjugate eigenvector elements of the beam lateral vibration. \( s_{b1} \) and \( s_{b2} \) are also complex conjugate eigenvalues of the beam lateral vibration which is shown as \( s_{1,2} \) in (28). \( E_{b3} \) and \( E_{b4} \) are complex conjugate eigenvector components which are the contribution of the longitudinal beam vibration to the lateral beam vibration. \( s_{b3} \) and \( s_{b4} \) are eigenvectors which are complex conjugate eigenvalues of the beam longitudinal vibration which are \( s_{3,4} \) in (28). \( E_{b5} \) and \( E_{b6} \) are complex conjugate eigenvector components which are the contribution of the torsional vibration to the lateral beam vibration. \( s_{b5} \) and \( s_{b6} \) are eigenvectors which are complex conjugate eigenvalues of the flexible shaft torsional vibration which are \( s_{5,6} \) in (28).

Similar to (31) equations for \( q_u(t) \) and \( q_s(t) \) can also be written. Equation (31) can be put in more familiar format by using algebra and trigonometric identities such as

\[ q_b(t) = X_{b1} e^{s_{b1} t} \sin\left( \omega_b \sqrt{1 - \zeta_b^2 t} + \Phi_{b1} \right) + \\ X_{b2} e^{s_{b2} t} \sin\left( \omega_b \sqrt{1 - \zeta_b^2 t} + \Phi_{b2} \right) + \\ X_{b3} e^{s_{b3} t} \sin\left( \omega_b \sqrt{1 - \zeta_b^2 t} + \Phi_{b3} \right) \]  

(32)

Here \( X_{b1} \) is the modal amplitude contributed by beam lateral vibration, \( \Phi_{b1} \) is the phase angle. \( X_{b2} \) is the modal amplitude contributed by the longitudinal vibration of the beam, \( \Phi_{b2} \) is the phase angle and \( X_{b3} \) is the modal amplitude contributed by the shaft vibration, \( \Phi_{b3} \) is the phase angle.

Similar equations can be written for \( q_u(t) \) and \( q_s(t) \) such as

\[ q_u(t) = X_{u1} e^{s_{u1} t} \sin\left( \omega_u \sqrt{1 - \zeta_u^2 t} + \Phi_{u1} \right) + \\ X_{u2} e^{s_{u2} t} \sin\left( \omega_u \sqrt{1 - \zeta_u^2 t} + \Phi_{u2} \right) + \\ X_{u3} e^{s_{u3} t} \sin\left( \omega_u \sqrt{1 - \zeta_u^2 t} + \Phi_{u3} \right) \]  

(33)

Here \( X_{u1} \) is the modal amplitude contributed by beam lateral vibration, \( \Phi_{u1} \) is the phase angle. \( X_{u2} \) is the modal amplitude contributed by the longitudinal vibration of the beam, \( \Phi_{u2} \) is the phase angle and \( X_{u3} \) is the modal amplitude contributed by the shaft vibration, \( \Phi_{u3} \) is the phase angle.
\[ q_i(t) = X_{ib} e^{-i\omega_b t} \sin(\omega_b \sqrt{1 - \frac{\zeta_i^2}{\zeta_i^2}} t + \Phi_{ib}) + \\
X_{ub} e^{-i\omega_b t} \sin(\omega_b \sqrt{1 - \frac{\zeta_i^2}{\zeta_i^2}} t + \Phi_{ub}) + \\
X_{sb} e^{-i\omega_b t} \sin(\omega_b \sqrt{1 - \frac{\zeta_i^2}{\zeta_i^2}} t + \Phi_{sb}) \]  

Fig. 5 shows modal amplitudes of the beam lateral vibration \( X_{ib} \), \( X_{ub} \) and \( X_{sb} \) with respect to the motor speed \( \omega_m \). To fit the three modal amplitudes in the same plot, their magnitudes are changed to \( dB \). Modal amplitudes and phase angles are calculated for shaft length of 0.1 m. For small values of the motor speed, the contribution of the longitudinal vibration to the lateral vibration of the beam is very high while the contribution of the shaft vibration is small. When the motor speed is increased, the magnitude of \( X_{ub} \) is decreased and the magnitude of \( X_{sb} \) is increased, but at the same time \( X_{ib} \) is also increased.

Fig. 6 shows the change of phase angles \( \Phi_{ib} \), \( \Phi_{ub} \) and \( \Phi_{sb} \) with respect to the motor speed. Phase angles stay almost constant but small change in the phase angles \( \Phi_{ib} \) and \( \Phi_{sb} \) can be seen around the motor speed 2000 to 2500 rpm.

Fig. 7 shows the modal amplitudes of the beam longitudinal vibration \( X_{ib} \), \( X_{ub} \) and \( X_{sb} \). The modal amplitude \( X_{ub} \) is not changing with the motor speed but \( X_{ub} \) decreases and \( X_{ub} \) increases approximately after motor speed 2000 rpm.

Fig. 8 shows the phase angles \( \Phi_{ib} \), \( \Phi_{ub} \) and \( \Phi_{sb} \) of the beam longitudinal vibration. The phase angle \( \Phi_{sb} \) is not changing by the motor speed but the beam lateral vibration phase angle \( \Phi_{sb} \) changes approximately from \(-90^\circ\) to \(+90^\circ\) after motor speed 1500 rpm.
Fig. 8 Phase angles of the beam longitudinal vibration with respect to motor speed

Fig. 9 shows the modal amplitudes of shaft torsional vibration which are $X_{bs}$, $X_{us}$ and $X_{ss}$. These modal amplitudes are not changing appreciably by the motor speed.

Fig. 10 shows the phase angles of the shaft torsional vibration. Similar to the modal amplitudes, the phase angles $\Phi_{bs}$, $\Phi_{us}$ and $\Phi_{ss}$ are also not changing with the motor speed.

Dynamic equation of the longitudinal vibration of beam has a forcing function on the right side, which is shown as $f_2$ in (26). This force is a centrifugal force that excites the vibrations of the system.

As an example Fig. 11, Fig. 12 and Fig. 13 are plotted to show the beam lateral and longitudinal vibrations and torsional vibrations for motor speed of 3000 rpm and shaft length of 0.1 m. Since the beam longitudinal vibration natural frequency is very high settling time of longitudinal vibration is relatively short, it damps out around 0.03 second.

Fig. 11 The beam lateral vibration for motor speed 3000 rpm
increases and remains equal to the motor speed $\omega_m$ until the motor speed is equal to the shaft-disk system natural frequency of $\omega_T$. Then the beam lateral natural frequency $\omega_b$ becomes equal to the natural frequency $\omega_T$ and stays same while the motor speed $\omega_m$ is increased. Since the system is coupled, each vibration is the combination of the three vibrations with modal amplitude values and phase angles. These modal amplitudes and phase angles are also plotted against the motor speed.

**REFERENCES**


Dr. Khaled Alnafie has been born in Taif, Saudi Arabia, 1967. He has obtained his Bachelor degree from the Department of Mechanical Engineering at King Abdulaziz University, Jeddah, Saudi Arabia in 1991. He has obtained his PhD in Mechanical Engineering from the University of Central Florida, Orlando, Florida, USA in 2000. He is currently an associate professor in the Department of mechanical engineering at King Abdulaziz University, Jeddah, Saudi Arabia and He is the Chairman of the Department. His research interests include rotor dynamics, robotics, modal analysis, vibrations, and damage detection.