On Detour Spectra of Some Graphs

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Abstract—The Detour matrix (DD) of a graph has for its \((i,j)\) entry the length of the longest path between vertices \(i\) and \(j\). The DD-eigenvalues of a connected graph \(G\) are the eigenvalues for its detour matrix, and they form the DD-spectrum of \(G\). The DD-energy \(E_{DD}\) of the graph \(G\) is the sum of the absolute values of its DD-eigenvalues. Two connected graphs are said to be DD-equienergetic if they have equal DD-energies. In this paper, the DD-spectra of a variety of graphs and their DD-energies are calculated.

Keywords—Detour eigenvalue (of a graph), detour spectrum (of a graph), detour energy (of a graph), detour - equienergetic graphs.

I. INTRODUCTION

Let \(G\) be a connected graph with vertex set \(V(G) = \{v_1, v_2, ..., v_n\}\). The ordinary graph spectrum is formed by the eigenvalues of the adjacency matrix \(A\). In what follows we denote the ordinary eigenvalues of the graph \(G\) by \(\lambda_i, i = 1, 2, ..., n\) and the respective spectrum by \(\text{spec}(G)\). The detour matrix \(DD(G)\) of \(G\) is defined so that its \((i,j)\) - entry is equal to the length of the longest path between vertices \(i\) and \(j\). The eigenvalues of the DD(G) and G form the DD-spectrum of \(G\), denoted by \(\text{spec}_{DD}(G)\). Since the detour matrix is symmetric, all its eigenvalues \(\mu_i, i = 1, 2, ..., n\) are real and can be labeled so that \(\mu_1 \geq \mu_2 \geq ... \geq \mu_n\). If \(\mu_1 \geq \mu_2 \geq ... \geq \mu_n\) are the distinct DD-eigenvalues, then the DD-spectrum can be written as

\[
\text{spec}_{DD}(G) = \{\mu_1, \mu_2, ..., \mu_n\}
\]

where \(m_j\) indicates the algebraic multiplicity of the eigenvalue \(\mu_j\). Of course, \(m_1 + m_2 + ... + m_n = n\). Two graphs \(G\) and \(H\) for which \(\text{spec}_{DD}(G) = \text{spec}_{DD}(H)\) are said to be DD-cospectral. Otherwise, they are non-DD-cospectral. The DD-energy, \(E_{DD}\) of \(G\) is defined as \(E_{DD} = \sum_{i=1}^{n} |\mu_i|\). Two graphs with equal DD-energy are said to be DD-equienergetic. DD-cospectral graphs are evidently DD-equienergetic. Therefore, in what follows we focus our attention on DD-equienergetic non-DD-cospectral graphs.

The concept of detour matrix was introduced in graph theory by F. Harary[6] for describing the connectivity in directed graphs. The detour matrix was then extensively studied in [9,8,10]. In the subsequent section we derive a Hoffman-type relation for the detour matrices of complete graphs, complete bipartite graphs and cycles. By means of it, the detour spectra of some graphs and their energies are obtained. The following results are used in the subsequent sections:

Result 1[4]. Let \(G\) be a graph with adjacency matrix \(A\) and \(\text{spec}(G) = \{\lambda_1, \lambda_2, ..., \lambda_n\}\). Then \(\det A = \prod_{i=1}^{n} \lambda_i\).

Result 2[5]. Let \(A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}\) be a \(2 \times 2\) block symmetric matrix. Then the eigenvalue of \(A\) are those of \(A_0 + A_1\) together with those of \(A_0 - A_1\).

Result 3 [4]. Let \(M, N, P\) and \(Q\) be matrices, and let \(M\) be invertible. Then

\[
\det S = \det(M)\det(Q - PM^{-1}N).
\]

Result 4 ([4]) \(\text{spec}(K_{n,n}) = \begin{bmatrix} n & 0 \\ 1 & 2n - 2 \end{bmatrix}\).

Result 5 ([7]) Let \(M\) be a real symmetric irreducible square matrix of order \(n\) in which each row sum is equal to a constant \(k\). Then there exists a polynomial \(Q(x)\) such that \(Q(M) = J\), where \(J\) is the all one square matrix whose order is the same as that of \(M\).

Result 6 ([7]). Let \(D\) be the distance matrix of a connected distance regular graph \(G\). Then \(D\) is irreducible and there exists a polynomial \(P(x)\) such that \(P(D) = J\). In this case \(D = \sum_{i=1}^{n} |\mu_i|\), where \(k\) is the unique sum of each row which is also the greatest simple eigenvalue of \(D\), whereas \(\lambda_2, \lambda_3, ..., \lambda_n\) are the other distinct eigenvalues of \(D\).

In [7], it is shown that the distance spectrum of double graph of any simple graph, Cartesian product of distance regular graph \(G\) with \(K_2\) and lexicographic product of any simple graph \(G\) with \(K_2\) depends on the distance spectrum of \(G\). However for detour spectrum this varies from graphs to graphs.

II. DETOUR SPECTRUM OF SOME GRAPHS

Theorem 2.1. If \(G\) is the complete graph of order \(n\), then the detour energy of \(G\) is \(E_{DD}(G) = 2(n - 1)^2\).

Proof. As the detour distance between any two disjoint vertices is \(n - 1\), it follows that \(DD(G) = (n-1)(J-I)\), where \(J\) is
Proof. The detour matrix
\[ D(G) = \begin{bmatrix}
3 & 1 \\
1 & n - 1
\end{bmatrix} \]
and consequently \( E_D(G) = (2n - 1)^2 \).

Theorem 2.2.
\[ Spec_{DD}(K_{n,n}) = \left\{ 4n^2 - 5n + 2, -(2n - 1), -(2n - 1) \right\} \]
where \( C = -(3n - 2) \).

Proof. The Theorem follows from the fact that \( DD(K_{n,n}) = (2n - 2) (I - J) + A \), where \( A \) is the adjacency matrix of \( K_{n,n} \) and from Result 4.

Corollary 3.2. \( E_{DD}(K_{n,n}) = 8n^2 - 10n + 4 \).

Theorem 3.4. If \( G \) is a cycle of length \( n \), then the largest eigenvalue of detour matrix of \( C \) is \( \mu_1 = n \).

Proof. Using Result 2, we get \( DD(G) = \begin{bmatrix}
2(n + n^2 - 1) & 2(n - 1) & -(2n - 1) \\
1 & 1 & n - 1
\end{bmatrix} \).

Theorem 3.5. If \( G \) is a cycle of length \( n \) with detour spectrum \( spec_{DD}(G) = \left\{ \mu_1, \mu_2, ..., \mu_n \right\} \), then
\[ spec_{DD}(DD(G)) = \begin{bmatrix}
2(n + n^2 - 1) & 2(n - 1) & -(2n - 1) \\
1 & 1 & n - 1
\end{bmatrix}, \quad i = 2, ..., n. \]

IV. THE DETOUR SPECTRUM OF THE CORONA OF G AND \( K_1 \)

Theorem 4.1. Let \( G \) be a connected detour regular graph with detour regularity \( k \). If \( spec_{DD}(G) = \{ \mu_1, \mu_2, ..., \mu_n \} \), then \( spec_{DD}(G \circ K_1) \) consists of numbers \( n + k - 1 + \sqrt{(n + k)^2 + (n - 1)^2}, n + k - 1 - \sqrt{(n + k)^2 + (n - 1)^2}, \mu_1 - 1 + \sqrt{\mu_1^2 + 1}, \mu_1 - 1 - \sqrt{\mu_1^2 + 1}, i = 2, ..., n \).

Theorem 5.1. If \( G \) is a cycle of length \( n \) with detour spectrum \( spec_{DD}(G) = \{ \mu_1, \mu_2, ..., \mu_n \} \), then \( spec_{DD}(DD(G)) = \begin{bmatrix}
2(n + n^2 - 1) & 2(n - 1) & -(2n - 1) \\
1 & 1 & n - 1
\end{bmatrix} \).

Corollary 5.4. \( E_{DD}(G \times K_2) = 8n^2 - 10n + 4 \).

The theorem now follows from Theorem 2.2.

Corollary 3.4. \( E_{DD}(D_2(K_{n,n})) = 8n^2 - 10n + 4 \).

Theorem 3.5. If \( G \) is a cycle of length \( n \) with detour spectrum \( spec_{DD}(G) = \{ \mu_1, \mu_2, ..., \mu_n \} \), then
\[ spec_{DD}(DD(G)) = \begin{bmatrix}
2(n + n^2 - 1) & 2(n - 1) & -(2n - 1) \\
1 & 1 & n - 1
\end{bmatrix}, \quad i = 2, ..., n. \]

Theorem 5.1. If \( G \) is the complete graph of order \( n \) with \( \mu_1 = n \), then
\[ spec_{DD}(DD(G)) = \begin{bmatrix}
2(n + n^2 - 1) & 2(n - 1) & -(2n - 1) \\
1 & 1 & n - 1
\end{bmatrix}. \]
Theorem 5.5. If G is the cycle of length n with
detour spectrum \( \{\mu_1 = r_1, \mu_2, ..., \mu_n\} \) and ordinary
spectrum \( \{\lambda_1 = r_1, \lambda_2, ..., \lambda_n\} \), then \( \text{spec}_{DD}(G) = \{\lambda_i = r_i, 2\mu_i - 2 - 2\lambda_i\} \).

Proof. Theorem follows from the fact that the
detour matrix of \( \text{DD}(G) \) has the form
\[
\begin{bmatrix}
2r_1 + (2n + 1)n - 2r_1 & 2\mu_1 - 2 - 2\lambda_1 & A & B \\
1 & 1 & 1 & 1
\end{bmatrix}
\]
where \( A = 2 - 3n + 2r_1 \) and \( B = 2(1 - n) + 2\lambda_i \).

VI. THE DETOUR SPECTRUM OF LEXICOGRAPHIC
PRODUCT OF SOME GRAPHS WITH \( K_2 \)

Theorem 6.1. If G is the complete graph of order n with
detour spectrum \( \text{spec}_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_n\} \) then
\( \text{spec}_{DD}(G[K_2]) = \)
\[
\begin{bmatrix}
4n^2 - 4n + 1 & -(2n - 1) & -(2n - 1) \\
1 & 1 & n - 1 \\
1 & 1 & n
\end{bmatrix}
\]

Proof. The detour matrix of \( G[K_2] \) is of the form
\[
\begin{bmatrix}
DD + (n - J - 1) & DD + (n - J + (n - 1)) & I \\
DD + (n - J + (n - 1)) & DD + (n - J - 1) & I \\
I & I & 1
\end{bmatrix}
\]
Using Result 2, we get
\[
\text{spec}_{DD}(G[K_2]) = \begin{bmatrix} 2(k + n - 1) & 2\mu_i - 1 & -(2n - 1) \\
1 & 1 & n \\
1 & 1 & n 
\end{bmatrix}, \quad i = 2, ..., n
\]
The theorem follows from Theorem 2.1.

Corollary 6.2. \( E_{DD}(G[K_2]) = (8n^2 - 8n + 2) \).

Theorem 6.3. If G is \( K_n \) with detour spectrum
\( \text{spec}_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_n\} \), then
\( \text{spec}_{DD}(G[K_2]) = \)
\[
\begin{bmatrix}
16n^2 - 8n + 1 & -(4n - 1) \\
1 & 4n - 1 \\
1 & 4n - 1 
\end{bmatrix}
\]

Proof. Since the detour matrix of \( G[K_2] \) is same as the detour
matrix of \( K_n \), the spectrum of \( G[K_2] \) is same as spectrum of
\( K_n \).

Corollary 6.4. \( E_{DD}(G[K_2]) = 32n^2 - 16n + 2 \).

Theorem 6.5. If G is a cycle of length n with detour spectrum
\( \text{spec}_{DD}(G) = \{\mu_1 = k, \mu_2, ..., \mu_n\} \) then
\( \text{spec}_{DD}(G[K_2]) = \)
\[
\begin{bmatrix}
4n^2 - 4n + 1 & -(2n - 1) \\
1 & 4n - 1 \\
1 & 4n - 1 
\end{bmatrix}
\]

Proof. Since the detour matrix of \( G[K_2] \) is same as the detour
matrix of \( K_n \), the spectrum of \( G[K_2] \) is same as spectrum of
\( K_n \).

Corollary 6.6. \( E_{DD}(G[K_2]) = 8n^2 - 8n + 2 \).

VII. THE EXTENDED DOUBLE COVER OF REGULAR
GRAPHS

In [1], N. Alon introduced the concept of extended double cover
graph of a graph as follows: Let G be a graph on the
vertex set \( \{v_1, v_2, ..., v_n\} \). Define a bipartite graph H with
\( V(H) = \{v_1, v_2, ..., v_n, u_1, u_2, ..., u_n\} \) in which \( v_i \) is adjacent
to \( u_i \) for each \( i = 1, 2, ..., n \) and \( v_i \) is adjacent to \( v_j \) in G.
The graph H is known as the extended double cover graph (EDC - graph) of G. The ordinary spectrum of H has been
determined in [3] and the distance spectrum of EDC of a
regular graph of diameter 2 has been determined in [7]. In
this section we obtain the detour spectrum of the EDC - graph
of a r - regular graph on n vertices.

Theorem 7.1. Let G be a r - regular graph on n vertices.
Then the DD- spectrum of the EDC- graph of G is given by
\[
\begin{bmatrix}
4n^2 - 5n + 2 & 2 - 3n & 2 - 2n \\
1 & 1 & 2n - 2 \\
1 & 1 & 2n - 2 
\end{bmatrix}
\]

Proof. For any regular graph G with vertices \( \{v_1, v_2, ..., v_n\} \) and
\( \{u_1, u_2, ..., u_n\} \) the new vertices to form the extended
double cover H, the detour path from \( v_i \) to \( u_j \) is
\( 2n - 1 \) and the detour path from \( v_i \) to \( v_j \) or \( u_i \) to \( u_j \) is
\( 2n - 2 \) for all \( i, j \). This shows that \( DD(H) = \)
\[
\begin{bmatrix}
2(2n - 2)(J - I) & (2n - 1)J \\
(2n - 1)J & (2n - 2)(J - I)
\end{bmatrix}
\]
Using Result 2 we get the required result.

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