

# On Method of Fundamental Solution for Nondestructive Testing

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**Abstract**—Nondestructive testing in engineering is an inverse Cauchy problem for Laplace equation. In this paper the problem of nondestructive testing is expressed by a Laplace's equation with third-kind boundary conditions. In order to find unknown values on the boundary, the method of fundamental solution is introduced and realized. Because of the ill-posedness of studied problems, the TSVD regularization technique in combination with L-curve criteria and Generalized Cross Validation criteria is employed. Numerical results are shown that the TSVD method combined with L-curve criteria is more efficient than the TSVD method combined with GCV criteria. The abstract goes here.

**Keywords**—ill-posed, TSVD, Laplace's equation, inverse problem, L-curve, Generalized Cross Validation.

## I. INTRODUCTION

**D**ETECTING the domain of corroding materials is an important topic in engineering and was introduced by Santosa[1]. In this study this mathematical model adopted is as follows[2]:

Let  $\Omega$  be a metallic body with constant conductivity and  $\partial\Omega$  be its boundary. Suppose that the corroding portion  $\Gamma_2$  is an inaccessible part of and portion  $\Gamma_1$  is an accessible portion of  $\partial\Omega$ . For simplification we restrict  $\Omega$  to the rectangular domain, where 'a' is a small constant, See Fig.1. This model was considered by Dario[2].

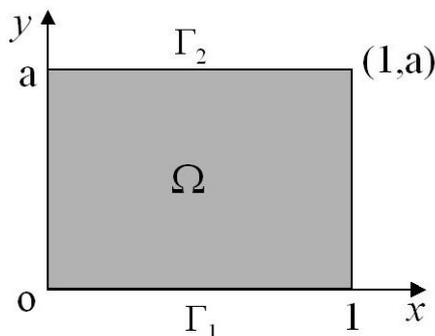


Fig. 1. domain  $\Omega$

If the domain  $\Omega$  is considered as the electrostatic field, the electric potential  $u$  satisfies the Laplace's equation in  $\Omega$ , i.e.,

$$\Delta u = 0, (x, y) \in \Omega. \quad (1)$$

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$$u_x(0, y) = u_x(1, y) = 0, y \in [0, a], \quad (2)$$

$$u_y(x, 0) = -\varphi(x), x \in [0, 1], \quad (3)$$

$$u_y(x, a) + \gamma(x)u(x, a) = 0, x \in [0, 1]. \quad (4)$$

Note that the non-negative function  $\gamma(x)$  in the third-kind boundary condition (4) is related to the corrosion rate at point  $x$  [2,3].

The forward problem of solving Laplace's equation is to get the potential  $u$  from the known data (2)-(4).

The inverse problem we discuss here is to recover the unknown function  $\gamma(x)$  from the known Cauchy data  $\varphi(x)$  and  $u(x, 0)$  on the accessible boundary  $\Gamma_1$  [2]. The inverse problem can be solved according to the following two steps:

**Step 1** Get the Cauchy data  $u_y(x, a)$  and  $u(x, a)$  on  $\Gamma_2$  from Cauchy problem for Laplace's equation:

$$\begin{cases} \Delta u = 0, (x, y) \in \Omega, \\ u(x, 0) = \varphi_1(x), x \in \Gamma_1, \\ u_n(x, 0) = \varphi_2(x), x \in \Gamma_1. \end{cases} \quad (5)$$

where the  $u_n(x, 0)$  is the outward normal derivative of  $u$  along  $\Gamma_1$ . The data  $\varphi_1(x), \varphi_2(x)$  can be obtained by measurements.

**Step 2** Recover the non-negative function  $\gamma(x)$  by (4). For such an inverse problem there exists a unique solution[4]. As shown in [2], the Galerkin method based on Fourier coefficients is proposed and tested. In order to get the positive approximations the maximum entropy principle is also discussed. The ideals of recovering corrosion can be viewed as a thermal imaging problem[5].

In this paper the MFS(The Method of Fundamental Solution) and regularization techniques are used to recover the non-negative function  $\gamma(x)$  in (4). It is well known that Cauchy problem (5) is typically ill-posed in the sense of Hadamard[6]. Most existing numerical methods such as finite element and finite difference often fail to produce an acceptable solution of (5). The reason of this phenomenon is that the Cauchy problem for Laplace's equation is very sensitive to the occurrence of measurement errors. Compared with the FEM and FDM, the MFS does not require the meshing of the domain  $\Omega$  and is easy to be operated. One disadvantage of MFS is that the resulting linear system is always ill-conditioned and some regularization techniques are needed.

This paper is organized as follows. In section 2 we establish a linear system by MFS and point out the necessary of regularization. In section 3 the TSVD regularization is used to solve the resulting ill-conditioned system. The truncated number of the SVD decomposition, which can be regarded as regularization parameter, can be determined by L-curve criteria and the GCV function. In section 4 we will give some

examples to show the efficiency of MFS combined with TSV regularization. In section 5 we give the conclusion.

## II. MFS AND REGULARIZATION

The MFS is a boundary-type radial basis function(RBF) collocation scheme. It was originally formulated by Kupradze and Aleksidze[7] and has been developed by numerous mathematicians and scientists over the past three decade. It poses integration-free,easy-to-use and meshfree merits. Denote

$$G(P, Q) = \frac{1}{2\pi} \ln \|P - Q\|$$

$$= \frac{1}{2\pi} \ln \sqrt{(x - x_Q)^2 + (y - y_Q)^2} \quad (6)$$

to be the fundamental solution of Laplace operator  $\Delta u = 0$ , where the points  $P(x, y)$  and  $Q(x_Q, y_Q) \in R^2$ . Particularly the source point  $Q(x_Q, y_Q) \in R^2$  is located outside the domain  $\Omega$ . The method of fundamental solution (MFS) can be expressed as follows.

On the accessible boundary  $\Gamma_1$ , we choose  $n$  collocation points  $P_i (i = 1, 2, \dots, n)$  to fit the Dirichlet data and  $m$  points  $P_i (i = n + 1, n + 2, \dots, n + m)$  to fit the Neumann data. The requirement on these collocation points are pairwise distinct. These nodes can be viewed as measure points and the measure data at these nodes may be affected by error. We also select a fictitious circle  $R$  containing  $\Omega$  in its interior and  $L$  source points  $Q_j (j = 1, 2, \dots, L)$  on  $R$ . See Fig.2.

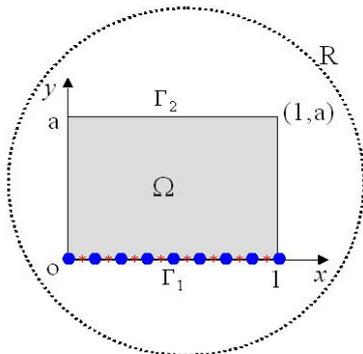


Fig. 2. collocation points . and \*

The method of fundamental solutions is based on the fact that an approximation  $\tilde{u}$  to the solution of (1) can be expanded by the basis functions  $G(P, Q_j), j = 1, 2, \dots, L$  as follows:

$$\tilde{u} = \sum_{j=1}^L a_j G(P, Q_j), \quad (7)$$

where  $a_j (j = 1, 2, \dots, L)$  are unknown coefficients. Note that the approximation solution  $\tilde{u}$  satisfies Laplace's equation (1) automatically but not the boundary conditions in (5). The unknown coefficients  $a_j$  of  $\tilde{u}$  must be selected carefully so that they exactly satisfy

$$\sum_{j=1}^L a_j G(P_i, Q_j) = \varphi_1(P_i),$$

$$\sum_{j=1}^L a_j \left. \frac{\partial G(P_i, Q_j)}{\partial n} \right|_{P=P_i} = \varphi_2(P_i). \quad (8)$$

where  $i = 1, 2, \dots, n$  and  $n + 1, \dots, n + m$ . The equation (8) can be written as a linear algebraic system equation

$$\begin{pmatrix} G(P_i, Q_j) \\ \frac{\partial G(P_i, Q_j)}{\partial n} \end{pmatrix}_{(n+m) \times L} \begin{pmatrix} a_1 \\ \vdots \\ a_L \end{pmatrix} = \begin{pmatrix} \varphi_1(P_i) \\ \varphi_2(P_i) \end{pmatrix}_{(n+m) \times 1}, \quad (9)$$

or in the form of

$$Ax = b \quad (10)$$

where the unknown vector  $x$  contains the coefficients  $a_j (j = 1, 2, \dots, L)$ . It should be noted that in order to uniquely determine the vector  $x$ , the number  $m + n$  must be greater or equal to the number of source points  $L$ . In this situations, The solution  $x$  of (10) can be determined by solving the following least squares problem

$$\min_{x \in R^L} \|b - Ax\|_2 \quad (11)$$

If we get the solution of (11), the function  $\tilde{u}(x, y)$  can be viewed as an approximation of the electric potential  $u(x, y)$ . The value of  $u(x, y)$  in the domain  $\Omega$  or along its boundary can be calculated from  $\tilde{u}(x, y)$ . Especially the Cauchy data can be obtained by  $\tilde{u}$  and the non-negative function  $\gamma(x)$  can be determined by (4).

As mentioned in section 1, the matrix  $A$  of (9) is ill-conditioned and the (9) is called discrete linear ill-posed system[8]. Because of the badly ill-conditioning of  $A$ , the solution of (11) is very sensitive to errors in  $b$  caused by measurement and the location of the source points. Most common methods such as LU- or QR-factorizations fail to produce an acceptable solution of (10). In order to compute a meaningful approximation  $x$ , regularization methods are required. The Tikhonov approach[9] is a well known method, i.e. replace the least squares problem (11) by

$$\min_{x \in R^L} \{ \|b - Ax\|_2 + \lambda^2 \|x\|_2 \} \quad (12)$$

where  $\lambda \in R$  is regularization parameter. In this paper we concentrate on the TSV method[10] combined with GCV and L-curve criteria to solve (10). Numerical examples show that this method can produce an acceptable non-negative function  $\gamma(x)$ .

## III. TSV REGULARIZATION

In this section we will discuss how to recover the function  $\gamma(x)$ . Supposed that the matrix  $A$  is decomposed into

$$A_{nm} = U_{nn} \Sigma_{nm} V_{mm}^T = \sum_{i=1}^m u_i \sigma_i v_i^T, \quad (13)$$

where  $U_{nn} = (u_1, u_2, \dots, u_n)$  and  $V_{mm} = (v_1, v_2, \dots, v_m)$  are column orthogonal matrices,  $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_m)$  is a diagonal matrix where the singular values,  $\sigma_i (i = 1, 2, \dots, m)$ , are arranged in descending order. The solution  $x^*$  of (10) using the SVD can be expressed as

$$x^* = \sum_{i=1}^m \frac{u_i^T b}{\sigma_i} v_i. \quad (14)$$

When the vector  $b$  is contaminated by measure errors, it is obvious that the solution  $x^*$  may be blown up by small singular value  $\sigma_i$  and the measure errors in  $b$ . Some regularization techniques should be introduced to filter out the parts of the solution corresponding to all the small singular values. By dropping off the small singular values, the Truncated SVD solution(TSVD) is defined as follows:

$$x_{reg} = \sum_{i=1}^r \frac{u_i^T b}{\sigma_i} v_i. \quad (15)$$

In (15) the truncated parameter  $r$  ( $r \leq m$ ) is called regularization parameter and it plays a major role in regularization. The so-called generalized cross validation (GCV)[11] and L-curve criteria[12] are two different ways of choosing the truncated parameter  $r$ . The advantage of these methods is that they do not require any knowledge about the noise lever presented in the right-hand side vector  $b$ . For completeness, we simply outline the GCV and L-Curve criteria.

#### A. GCV criteria

Let  $x_{reg} = A^\# b$  denote the solution computed by means of (15), the regularization parameter  $r$  lets the following GCV function

$$GCV(r) = \frac{\|(I - AA^\#)b\|^2}{[\text{trace}(I - AA^\#)]^2}, r = 1, 2, \dots, m \quad (16)$$

minimize with regard to  $r$ . It is easy to rewrite the GCV function in terms of the SVD of  $A$  as follows[13]:

$$GCV(r) = \sum_{i=r+1}^m |u_i^T b|^2 / (n - r)^2, r = 1, 2, \dots, m.$$

#### B. L-curve criteria

The L-curve criteria is an other popular method for choosing a suitable value of the parameter  $r$ . When the 2-norm of the solution  $x_{reg}$  and the residual  $b - Ax_{reg}$  are known, the curve  $(\log \|x_{reg}\|, \log \|b - Ax_{reg}\|)$ ,  $r = 1, 2, \dots, m$  is usually referred to the L-curve. The generic shape of the L-curve is shown in Fig.3.

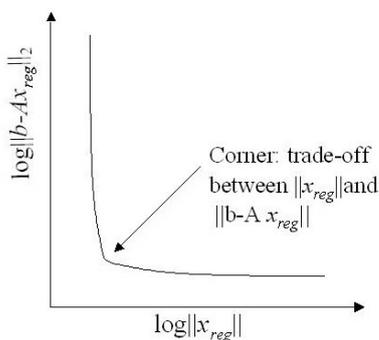


Fig. 3. L-curve for a continuous regularization parameter

The L-curve criterion defines the optimal value of the regularization parameter  $r$  to be at the corner of the L-curve.

If the optimal regularization parameter  $r$  is determined by L-curve criterion or GCV criterion, then the approximation

$u = \sum_{j=1}^L a_j G(P, Q_j)$  can be obtained by solving (11) and the non-negative function  $\gamma(x)$  can be recovered by (4).

The complete algorithm for recovering  $\gamma(x)$ , including determination of the regularization parameter  $\lambda$ , takes the following form:

#### Algorithm (TSVD for recovering $\gamma(x)$ )

- (1) Set  $n, m, L$  and the radius of the fictitious circle  $R$ ;
- (2) Form linear system (10) by fundamental solution (6);
- (3) Solve (10) by TSVD method, where the regularization parameter  $r$  is determined by GCV criteria or L-Curve criteria;
- (4) Use (7) to calculate the Cauchy data  $u_y(x, a)$  and  $u(x, a)$  along the inaccessible boundary  $\Gamma_2$ ;
- (5) recover  $\gamma(x)$  from (4).

### IV. NUMERICAL RESULTS

In this section two examples are tested to illustrate the performance of the algorithm described above. For simplicity, we assume that the coefficient ' $a$ ' of domain  $\Omega$  is 0.1. The Neumann data  $\phi(x)$  in (3) is equal to 1. For numerical error estimation, we define the relative error between the recovered  $\tilde{\gamma}(x)$  and the exact  $\gamma(x)$  by the following formula

$$\varepsilon = \frac{\|\gamma(x) - \tilde{\gamma}(x)\|}{\|\gamma(x)\|}.$$

In order to test the influence of the noise, we add the simulated random white noises, i.e. normally distributed random entries with zero mean and the variance is chosen to be  $\delta$ , to the Cauchy data along  $\Gamma_1$ . Different proportional noise lever  $\delta$  will be considered here. We hope that the noises can be effectively filtered out by regularization method. In the following figures, "TSVD+L-Curve" represents that the corrosion rate  $\tilde{\gamma}(x)$  is computed by L-Curve criteria and "TSVD+GCV" represents that the  $\tilde{\gamma}(x)$  is computed GCV criteria. The solid line represents the exact  $\gamma(x)$  and the dotted line represents the approximation  $\tilde{\gamma}(x)$ .

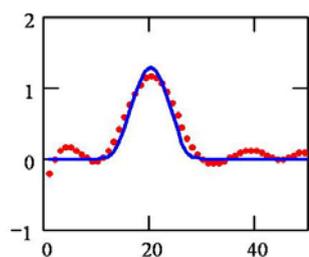
Example 1[2] Let

$$\gamma(x) = \begin{cases} \frac{10^6}{2}(x - 0.2)^4(x - 0.6)^4, & 0.2 \leq x \leq 0.6; \\ 0, & \text{other.} \end{cases}$$

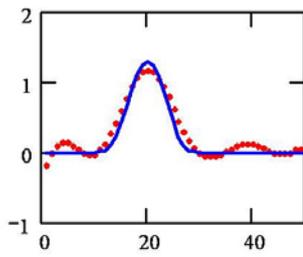
be the exact recovered solution of problem (1)-(4). The Cauchy data  $\varphi_1(x_i) = u_n(x_i, 0)$  ( $i = 1, 2, \dots, m$ ) and  $\varphi_2(x_j) = u_n(x_j, 0)$  ( $j = 1, 2, \dots, m$ ) along  $\Gamma_1$  can be obtained from problem (1)-(4). Here  $n = m = 100, L = 50$  and the radius of the fictitious circle  $R$  is 5.5. See Fig.2. The function  $\gamma(x)$  can be recovered from (4). For different error lever  $\delta$ , the comparison between the recovered  $\tilde{\gamma}(x)$  and the exact  $\gamma(x)$  are listed in the following figures.

Example 2 Consider the harmonic function  $u(x, y) = y^3 - 3yx^2 + x^2 - y^2 + 6$ . The method of simulating Cauchy data along  $\Gamma_1$  is the same as Example 1. For different error lever  $\delta$ , the comparison between the recovered  $\tilde{\gamma}(x)$  and the exact  $\gamma(x)$  is plotted in the following figures.

From these examples we find that the curves of recovered  $\tilde{\gamma}(x)$  are almost the same as that of the  $\gamma(x)$ , except for the curve computed by GCV criteria in Fig.13. The efficiency of L-Curve method is better than the GCV's because its relative

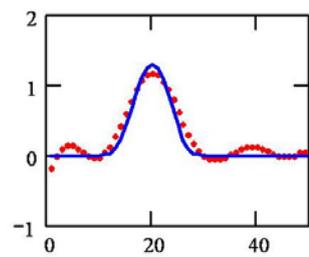


(a) "TSVD+L-Curve",  $\epsilon = 0.24$

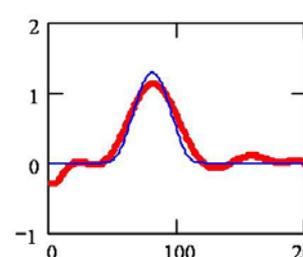


(b) "TSVD+GCV",  $\epsilon = 0.26$

Fig. 4. The results of applying the algorithm to example 1 with  $\delta = 10^{-4}$ ,

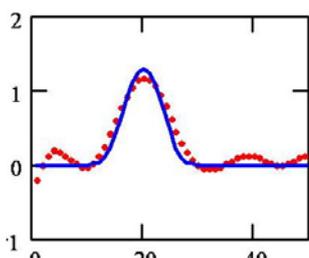


(a) "TSVD+L-Curve",  $\epsilon = 0.23$

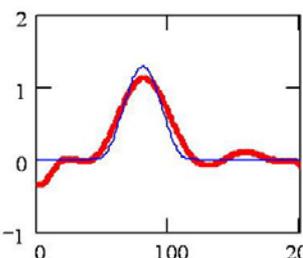


(b) "TSVD+GCV",  $\epsilon = 0.25$

Fig. 5. The results of applying the algorithm to example 1 with  $\delta = 10^{-3}$ ,

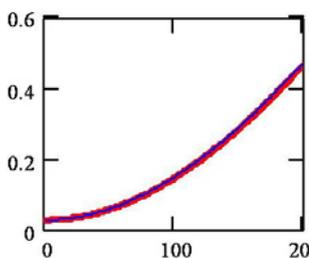


(a) "TSVD+L-Curve",  $\epsilon = 0.242$

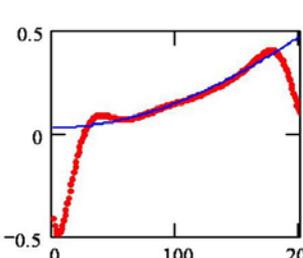


(b) "TSVD+GCV",  $\epsilon = 0.26$

Fig. 6. The results of applying the algorithm to example 1 with  $\delta = 0$ ,



(a) "TSVD+L-Curve",  $\epsilon = 0.000394$



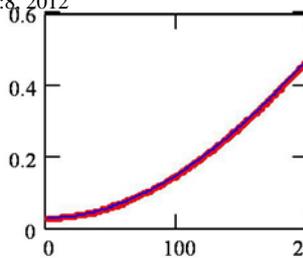
(b) "TSVD+GCV",  $\epsilon = 0.0576$

Fig. 7. The results of applying the algorithm to example 2 with  $\delta = 10^{-3}$ ,

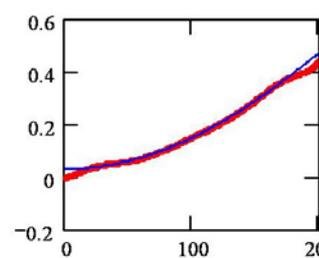
errors for different  $\delta$  are relatively smaller than those of the GCV's.

## V. CONCLUSION

In this paper the MFS combined with regularization was introduced to recover the  $\gamma(x)$  from the Cauchy problem of

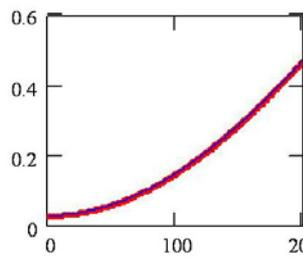


(a) "TSVD+L-Curve",  $\epsilon = 0.000394$

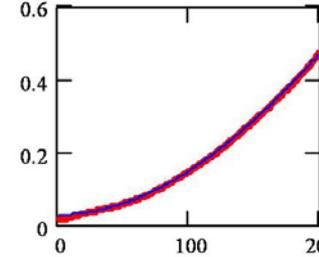


(b) "TSVD+GCV",  $\epsilon = 0.0576$

Fig. 8. The results of applying the algorithm to example 2 with  $\delta = 10^{-3}$ ,



(a) "TSVD+L-Curve",  $\epsilon = 0.000064$



(b) "TSVD+GCV",  $\epsilon = 0.0085$

Fig. 9. The results of applying the algorithm to example 2 with  $\delta = 10^{-4}$ ,

Laplace's equation. The limited numerical experience demonstrated that the developed method is numerically stable. Numerical results also showed that the TSVD combined with L-curve criteria was slightly better than the method combined with GCV criteria.

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