Some properties of b-weakly compact operators on Banach lattice

Na Cheng and Zi-li Chen

Abstract—We investigate the sufficient condition under which each positive b-weakly compact operator is Dunford-Pettis. We also investigate the necessary condition on which each positive b-weakly compact operator is Dunford-Pettis. Necessary condition on which each positive b-weakly compact operator is weakly compact is also considered. We give the operator that is semi-compact, but it is not b-weakly. We present a necessary and sufficient condition under which each positive semi-compact operator is b-weakly compact.

Keywords—b-weakly compact, Dunford-Pettis operator, M-weakly compact operator, L-weakly compact operator, semi-compact operator, weakly sequentially continuous lattice operations, order continuous norm, positive Schur property.

I. INTRODUCTION

RECALL that a subset $A$ of a Riesz space $E$ is called b-order bounded in $E$ if it is order bounded in $(E^\sim)^\sim$. A Riesz space is said to have property (b) if $A \subseteq E$ is order bounded whenever $A$ is order bounded in $(E^\sim)^\sim$. Note that every perfect Riesz space and therefore every order dual has property (b). Every reflexive Banach lattice has property (b). Every KB space has property (b) and if $(E^\sim)^\sim$ is retractive on $E$ then $E$ has property (b). On the other hand, by considering $A = \{e_n\}$ in $c_0$, we see that $c_0$ does not have property (b). An operator $T : E \to X$, mapping each b-order bounded subset of Banach lattice $E$ into a relatively weakly compact subset of Banach space $X$ is called a b-weakly compact operator. The collection of b-weakly compact operators will be denoted by $W_b(E, F)$. Then $W_b(E, F)$ is a closed subspace of $L(E, F)$, the vector space of all continuous operators from $E$ into $F$. Operators mapping order intervals into relatively weakly compact sets are called order-weakly operators and denoted by $W_n(E, F)$. The collection of weakly compact operators will be denoted by $W(E, F)$. Then $W(E, F) \subseteq W_b(E, F) \subseteq W_n(E, F)$,[9] gave examples to show that these inclusions may be proper.

An operator is said to be a Dunford-Pettis operator if it carries relatively weakly compact subsets onto norm totally bounded subsets. An operator $T$ from a Banach lattice $E$ into a Banach lattice $F$ is said to be M-weakly compact if each disjoint bounded sequence $(x_n)$ of $E$, we have $\lim_n \|T(x_n)\| = 0$. And an operator $T$ from a Banach lattice $E$ into a Banach lattice $F$ is called L-weakly compact if for each disjoint bounded sequence $(y_n)$, in the solid hull of $T(B_E)$, we have $\lim_n \|y_n\| = 0$ where $B_E$ is the closed unit ball of $E$.

In 2003, S.Alpay and B.Altin [9] studied the property (b). They proved that Banach lattice $E$ is a KB-space if and only if it has order continuous norm and property (b) [9, Theorem 2.1]. They also gave the definition of b-weakly compact operator. They characterized that $T : E \to X$ is b-weakly compact operator if and only if for each b-order bounded $A \subseteq E$ and disjoint sequence $(x_n)$ in $A$ satisfies $\lim_n \|T(x_n)\| = 0$ [9, Theorem 2.8]. In 2006, S.Alpay and B.Altin [10] investigate Riesz spaces and Banach lattices enjoying property (b). They proved that if Banach $F$ is Dedekind complete, then the space of order bounded operators from Banach $E$ into $F$ has property (b) if and only if $E$ has property (b) [10, Theorem 2]. Every order closed Riesz subspace of a Dedekind complete Riesz space $E$ with property (b) has property (b) [10, Theorem 2]. In 2007, S.Alpay and B.Altin [11] characterized the b-weak compactness of $T$ in terms of its mapping properties [11, Theorem 1, Theorem 2, Theorem 4]. In 2007, B.Altin [13] investigated the order structure of b-weakly compact operator. In 2009, S.Alpay and B.Altin [12] gave characterized of KB-spaces in terms of b-weakly compact operators. A Banach lattice $F$ is KB-space if and only if for each Banach lattice $E$ and positive disjointness preserving operator $T : E \to F$ is b-weakly compact. In 2009, B. Aqzzouz and A. Elbou, and J. Hmichane [14] establish necessary and sufficient conditions under which b-weakly compact operators between Banach lattices have b-weakly compact adjoint or operators with b-weakly compact adjoint are themselves b-weakly compact.$T : E \to F$ between Banach lattices is a b-weakly compact operator, then its adjoint $T^\prime : F^\prime \to E^\prime$ is b-weakly compact if and only if $F^\prime$ or $E^\prime$ is a KB-space. Each operator $T : E \to F$ is b-weakly compact whenever its adjoint $T^\prime : F^\prime \to E^\prime$ is b-weakly compact if and only if $E$ or $F$ is a KB-space.

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is an AM-space if and only if the norm is additive on the positive cone of the dual. An element $\varepsilon > 0$ in a Riesz space is said to be an order unit whenever for each $x$ there exists some $\lambda > 0$ with $|x| \leq \lambda e$. Now if a Banach lattice $E$ has an order unit $e > 0$, then $\Lambda_e = E$ holds, and the norm $\|x\|_\infty = \inf\{|\lambda > 0 : |x| \leq \lambda e\}$ is equivalent to the original norm of $E$. In other words, if a Banach lattice $E$ has an order unit $e$, then $E$ can be renormed in such a way that it becomes an AM-space having $[-e, e]$ as its closed unit ball. A Banach lattice has order continuous norm if and only if every order bounded disjoint sequence is norm convergent to zero. A Banach lattice $E$ is said to be a KB-space, whenever every increasing norm bounded sequence of $E^+$ is norm convergent. For example, each reflexive Banach lattice is KB-space. Also, each KB-space has an order continuous norm, but a Banach lattice with an order continuous norm is not necessary a KB-
space. In fact, the Banach lattice \( c_0 \) has an order continuous norm but it is not a KB-space. However, if \( E \) is a Banach lattice, the topological dual \( E' \) is a KB-space if and only if its norm is order continuous. The Banach lattice \( E \) has the positive Schur property if each weakly null sequence with positive sequence in \( E \) converges to zero in norm. A Banach lattice \( E \) is said to have weakly sequentially continuous lattice operations whenever \( x_n \xrightarrow{w} 0 \) in \( E \) implies \( \|x_n\| \xrightarrow{w} 0 \) in \( E \). In an AM-space the lattice operations are weakly sequentially continuous. Also, every Banach lattice with the Schur property (i.e., \( x_n \xrightarrow{w} 0 \) implies \( \|x_n\| \xrightarrow{w} 0 \)) has weakly sequentially continuous lattice operations. Thus, for example, the Banach lattice \( C[0,1], l_1, l_1 \oplus C[0,1] \) all have weakly sequentially continuous lattice operations.

The goal of this paper is to investigate the sufficient condition under which each positive b-weakly compact operator is Dunford-Pettis. We also investigate the necessary condition on which each positive b-weakly compact operator is Dunford-Pettis. Necessary condition on which each positive b-weakly compact operator is weakly compact is also considered. We give the operator that is semi-compact, but it is not b-weakly compact. We present a necessary and sufficient condition under which each positive semi-compact operator is b-weakly compact.

All notions concerning Banach lattices and not explained here are can find in [1] and [2].

II. PROPERTIES OF B-WEAKLY COMPACT OPERATORS

**Theorem 1:** For Banach lattice \( F \), each positive b-weakly compact operator from AM-space into \( F \) is Dunford-Pettis.

**Proof:** Let \( \rho(x) = \|Tx\| \) for every \( x \in E \), then \( \rho \) is a continuous lattice seminorm on \( E \). Suppose \( T : E \rightarrow F \) is not a Dunford-Pettis operator, since AM-space has weakly sequentially continuous lattice operations, there exists a sequence \( \{x_n\} \subset E_+ \) with \( x_n \xrightarrow{w} 0 \) and \( \|Tx_n\| \geq 1 \).

Corollary 2.3.5 of [2] shows that for every \( 0 < c < 1 \), there exists a subsequence \( k(n) \in N \) and a disjoint sequence \( \{y_n\} \subset E_+ \) such that

\[
y_n \leq x_{k(n)}, \|Ty_n\| \geq c
\]

for all \( n \in N \). Since \( y_n \leq x_{k(n)} \) and \( x_n \xrightarrow{w} 0 \), the uniform boundedness theorem implies that the sequence \( y_n \) is bounded. Observing that \( (y_1 + \cdots + y_n) \xrightarrow{n} 0 \) is a monotone norm bounded sequence, there exists \( x' \in E_+ \) such that

\[
0 \leq y_1 + \cdots + y_n \leq x'
\]

together with the fact that \( T \) is b-weakly compact, it follows that

\[
\|Ty_n\| \rightarrow 0 (n \rightarrow \infty)
\]

This gives a contradiction. \( \square \)

**Theorem 2:** Let \( E \) and \( F \) be two Banach lattices, if every positive b-weakly compact operator \( T : E \rightarrow F \) is Dunford-Pettis, then the norm of \( F \) is order continuous or the lattice operations of \( E \) are weakly sequentially continuous.

**Proof:** If the norm of \( F \) is not order continuous and the lattice operations of \( E \) are not weakly sequentially continuous, A.W.Wickstead constructed in the proof of Theorem 2 of [4] two positive operators \( S, T : E \rightarrow F \) such that \( 0 \leq S \leq T \) and \( T \) is compact and hence it is b-weakly compact, Proposition 2.2 of [6] implies \( S \) is b-weakly compact, but it is not Dunford-Pettis. \( \square \)

**Theorem 3:** Let \( E \) and \( F \) be two Banach lattices, if every positive b-weakly compact operator \( T : E \rightarrow F \) is weakly compact, then one of the following statements is valid:

(a) The norm of the topological dual \( E' \) is order continuous.

(b) \( F \) is reflexive.

**Proof:** Suppose that neither the norm of \( E' \) is order continuous nor \( F \) is reflexive, then there exist a sublattice \( H \) of \( E \) which is isomorphic to \( l_1 \) and a positive projection \( P : E \rightarrow l_1 \).

On the other hand, since the closed unit ball \( B_F \) of \( F \) is not weakly compact, there exists a sequence \( (e_n)_n \) in \( B_F \) which does not have any weakly convergent subsequence.

Consider the operator \( S : l_1 \rightarrow F \) defined by

\[
S(x_n) = \sum_{n=1}^{\infty} x_n e_n
\]

It is easy to see that \( S \) is a KB-space, it is b-weakly compact, but it is not weakly compact. \( \square \)

**Theorem 4:** Let \( E \) and \( F \) be two Banach lattices, if each positive o-weakly compact operator \( T : E \rightarrow F \) is L-weakly compact, then one of the following conditions holds.

(a) \( F \) are KB-spaces.

(b) \( E' \) has the positive Schur property.

**Proof:** Suppose \( F \) is not a KB-space, Theorem 2.4.12 of [4] implies that \( F \) contains a sublattice isomorphic to \( c_0 \). Applying Theorem 3.1 of [3] it suffices to show each disjoint weak null sequence \( (x'_n)_n \subset E'_+ \) is norm convergent to 0.

For each \( x \in E \) define \( T : E \rightarrow c_0 \) by

\[
Tx = (x'_n(x))_n
\]

Theorem 17.5 of [1] implies \( T \) is a weakly compact operator, hence it is o-weakly compact, it is L-weakly compact. Theorem 18.13 of [1] implies

\[
T' : l_1 \rightarrow E'
\]

is M-weakly compact. As

\[
T'(e_n) = x'_n
\]

for all \( n \in N \), where \( e_n \) is the sequence with \( n \)’th entry equals to 1 and all others are zero, we conclude that

\[
\|x'_n\| \rightarrow 0 (n \rightarrow \infty)
\]

\( \square \)

Recall that a continuous operator \( T : E \rightarrow F \) is said to be semi-compact if for each \( \epsilon > 0 \), there exists some \( u \in \mathbb{F}^+ \) such that \( T(U) \subset [-u, u] + \epsilon V \) where \( U, V \) denote the closed unit balls of \( E \) and \( F \), respectively. Each compact operator, M-weakly compact (L-weakly compact) operator between Banach lattice is semi-compact. However, a semi-compact operator need not be compact, weakly compact, M-weakly compact (L-weakly compact). For instance, the identity operator \( I : \ell_\infty \rightarrow \ell_0 \) is semi-compact, but it does not have any one of the above mentioned compactness properties.
Theorem 5: Let E and F be nonzero Banach lattices such that F is \(\sigma\)-Dedekind complete. Then the following statements are equivalent.

1) Each positive semi-compact operator \(T : E \to F\) is b-weakly compact.
2) At least one of the following conditions holds:
   a) The norm of E is order continuous.
   b) The norm of F is order continuous.

Proof: 2) – a) \(\Rightarrow\) 1) Suppose that E has order continuous norm and \(T : E \to F\) is a positive semi-compact operator. Theorem 12.9 of [1] implies that each order interval of Banach lattice E is weakly compact, together with the fact that T is a positive semi-compact operator, it follows that T is weakly compact. Hence, T is b-weakly compact.

2) – b) \(\Rightarrow\) 1) Suppose that F has order continuous norm and \(T : E \to F\) is a positive semi-compact operator. For each \(\epsilon > 0\) there exists some \(u \in F^+\) such that
\[
T(U) \subseteq [-u, u] + \epsilon V
\]

U and V denote the closed unit balls of E and F, respectively. Theorem 12.9 of [1] implies that the order interval \([-u, u]\) in F is weakly compact, combined with Theorem 10.17 of [1] show that \(T(U)\) is relatively weakly compact, it follows that T is weakly compact. Hence, T is b-weakly compact.

1) \(\Rightarrow\) 2) Assume by way of contradiction that neither E nor F has an order continuous norm. To finish the proof, we have to construct a positive semi-compact operator \(T : E \to F\) which is not b-weakly compact.

Since the norm on E is not order continuous, applying Theorem 12.13 of [1] that there exists some \(x \in E^+\) and a sequence \((x_n)\) \(\subset [0, y]\) which does not converge to zero in norm. We may assume that \(\|x_n\| = 1\) for all \(n\).

Hence, by lemma 2.1 of [15] there exists a positive disjoint sequence \((g_n)\) of \(E^+\) with \(\|g_n\| \leq 1\) such that
\[
g_n(x_n) = 1 \quad \text{for all} \quad n \quad \text{and} \quad g_n(x_m) = 0 \quad \text{for} \quad n \neq m.
\]

For all \(x \in E\), define the positive operator \(R : E \to \ell_\infty\) by
\[
R(x) = (g_1(x), g_2(x), \cdots)
\]

Note that \(R(B_E) \subset B_{\ell_\infty}\).

On the other hand, as the norm on F is not order continuous, applying Theorem 12.13 of [1] that there exists some \(y \in F^+\) and a sequence \((y_n)\) \(\subset [0, y]\) which does not converge to zero in norm. We may assume that \(\|y_n\| = 1\) for all \(n\).

Since \(\sum_{i=1}^{\infty} y_i \leq y\) holds for all \(n\), and F is \(\sigma\)-Dedekind complete, for all \((\alpha_1, \alpha_2, \cdots) \in \ell_\infty\), define the positive operator \(S : \ell_\infty \to F\) by
\[
S(\alpha_1, \alpha_2, \cdots) = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i y_i
\]

Defines a lattice isomorphism from \(\ell_\infty\) into F where \(\lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i y_i\) denotes the order limit of the partial sum \(\sum_{i=1}^{n} \alpha_i y_i\).

Since the sequence \((y_n)\) is order bounded and disjoint, for each \((\alpha_1, \alpha_2, \cdots) \in B_{\ell_\infty}\), we see that
\[
|S(\alpha_1, \alpha_2, \cdots)| = \lim_{n \to \infty} \sum_{i=1}^{n} |\alpha_i| y_i \leq (\sup |\alpha_i|) \cdot y \leq y
\]

Hence \(S(\alpha_1, \alpha_2, \cdots) \in [-y, y]\), and we have \(S(B_{\ell_\infty}) \subset [-y, y]\).

Now consider the operator \(T = S \circ R : E \to F\) by
\[
T(x) = \lim_{n \to \infty} \sum_{i=1}^{n} g_i(x)y_i
\]

it is positive, and we have
\[
T(B_E) = S(R(B_E)) \subset S(B_{\ell_\infty}) \subset [-y, y]
\]

It is clear that T is semi-compact.

On the other hand, for all \(n\), we have
\[
T(x_n) = \lim_{n \to \infty} \sum_{i=1}^{n} g_i(x_n)y_i = y_n
\]

It follows that \(\|T(x_n)\| = \|u_n\| = 1\). As the sequence \((x_n)\) is order bounded and disjoint in E, it is clear that T is not order weakly compact. Hence, T is not b-weakly compact.

\[\square\]

Theorem 6: Let E and F be nonzero Banach lattices. Then the following statements are equivalent.

1) Each positive semi-compact operator \(T' : F' \to E'\) is b-weakly compact.
2) At least one of the following conditions holds:
   a) The norm of \(E'\) is order continuous.
   b) The norm of \(F'\) is order continuous.

Proof: 1) \(\Rightarrow\) 2) Assume by way of contradiction that neither \(E'\) nor \(F'\) has an order continuous norm. To finish the proof, we have to construct a positive semi-compact operator \(T' : F' \to E'\) which is not b-weakly compact.

Since the norm on \(E'\) is not order continuous, applying Theorem 2.6 of [15] that there exists a disjoint sequence \(\{x_n\} \subset E^+\) with \(\|x_n\| \leq 1\) for all \(n\) and there exists some \(0 \leq x' \in E'\) with \(x'(x_n) = 1\) for all \(n\). Moreover, the components \(x'_n\) of \(x'\), in the carrier \(C_{x_n}\) from an order bounded disjoint sequence in \((E')^+\) such that
\[
x'_n(x_n) = x'(x_n) = 1 \quad \text{for all} \quad n \quad \text{and} \quad x'_n(x_m) = 0 \quad \text{for} \quad n \neq m.
\]

Note that \(0 \leq x'_n \leq x'\) holds for all \(n\).

For all \(x \in E\), define the positive operator \(R : E \to \ell_1\) by
\[
R(x) = (x'_n(x))^\infty_{n=1}
\]

Since \(\sum_{i=1}^{\infty} |x'_n(x)| \leq \sum_{i=1}^{\infty} x'_n(|x|) \leq x'(|x|)\) holds for each \(x \in E\), the operator R is well defined.

On the other hand, as the norm on \(F'\) is not order continuous, applying Theorem 12.13 of [1] that there exists some \(f' \in F'^+_\ast\) and a disjoint sequence \((f'_n) \subset [0, f']\) which does not converge to zero in norm. We may assume that \(\|f'_n\| = 1\) for all \(n\). Hence, for each \(n\), we can choose \(f_n \in F^+\) with \(\|f_n\| = 1\) and \(f'_n(f_n) \geq \frac{1}{2}\|f'_n\| = \frac{1}{2}\).

For all \((\lambda_n) \in \ell_\infty\) consider the positive operator \(S : \ell_\infty \to F'\) defined by
\[
S(\lambda_n) = \sum_{n=1}^{\infty} \lambda_n f_n
\]

Since \((\lambda_n) \in \ell_\infty\) and \(\sum_{n=1}^{\infty} \|\lambda_n f_n\| = \sum_{n=1}^{\infty} |\lambda_n|\), it follows that S is well defined.
Now, for all $x \in E$, consider the operator $T = S \circ R : E \to F$ defined by

$$T(x) = \sum_{n=1}^{\infty} x_n f_n$$

Its adjoint $T' : F' \to E$ defined by

$$T'(g') = \sum_{n=1}^{\infty} g'_n(f_n)x'_n$$

for all $g' \in F'$. Since $\ell_\infty$ is an AM-space with unit, it follows that $R'$ is semi-compact, hence $T'$ is semi-compact.

On the other hand, note that the sequence $f'_n$ is order bounded and disjoint, and

$$\|T(f'_n)\| = \|\sum_{i=1}^{\infty} f'_n(f_n)x'_n\|$$

$$\geq \|f'_n(f_n)x'_n\| \geq \frac{1}{2} \|x'_n\|$$

$$\geq \frac{1}{2} x'_n(x_n) \geq \frac{1}{2}$$

Hence, $T'$ is not o-weakly compact, it is not b-weakly compact.

III. CONCLUSIONS

In this paper, we investigate the sufficient condition under which each positive b-weakly compact operator is Dunford-Pettis. We also investigate the necessary condition on which each positive b-weakly compact operator is Dunford-Pettis. Necessary condition on which each positive b-weakly compact operator is weakly compact is also considered. We give the operator that is semi-compact, but it is not b-weakly. We present a necessary and sufficient condition under which each positive semi-compact operator is b-weakly compact.

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