Transient Thermal Stresses of Functionally Graded Thick Hollow Cylinder under the Green-Lindsay Model

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Abstract—The transient thermoelastic response of thick hollow cylinder made of functionally graded material under thermal loading is studied. The generalized coupled thermoelasticity on the Green-Lindsay model is used. The thermal and mechanical properties of the functionally graded material are assumed to be varied in the radial direction according to a power law variation as a function of the volume fractions of the constituents. The thermal and elastic governing equations are solved by using Galerkin finite element method. All the finite element calculations were done by using commercial finite element program FlexPDE. The transient temperature, radial displacement, and thermal stresses distribution through the radial direction of the cylinder are plotted.

Keywords—Finite element method; Thermal stresses; Green-Lindsay theory; Functionally graded material.

I. INTRODUCTION

FUNCTIONALLY graded material (FGM) is nonhomogeneous material classified by its graded structure. The thermal and mechanical properties of the FGM are spatially varying designed to optimize the performance under thermal and mechanical loading.

FGM is designed to reduce the thermal stresses induced in the mechanical elements that are subjected to high temperatures. The FGM usually contains ceramic and metal constituents and the composition is varied as a function of the volume fractions of these constituents.


The objective of this paper is to study the dynamic thermoelastic response of FGM hollow circular cylinder considering the Green-Lindsay (GL) model. The thermal and mechanical properties of the FGM considered in this paper are assumed to be independent of temperature and varied in the radial direction using a power law variation as a function of the volume fractions of the material constituents. The coupled transient governing equations for the temperature and displacement fields in a FGM cylinder considering the GL theory are derived and solved. These equations are solved numerically by using Galerkin finite element method.

II. GOVERNING EQUATIONS

The transient heat conduction equation for FGM as deduced from the generalized thermoelastic theories [12] without heat generation is

\[ (kT_x)_t + \tau_\alpha \frac{\partial (kT_x)}{\partial t} = \rho c_x (\dot{T} + \tau_\epsilon \frac{\partial \dot{T}}{\partial t} + (3\lambda + 2\mu) \alpha T_x (\dot{\epsilon}_{xx} + \tau_\epsilon \frac{\partial \dot{\epsilon}_{xx}}{\partial t}) \]

where \( k \) is the thermal conductivity, \( \rho \) the density, \( c_x \) the specific heat at constant strain, \( T \) the absolute temperature, \( \dot{\epsilon}_{xx} \) the strain tensor, \( \lambda \) and \( \mu \) the Lamé constants, \( \alpha \) the thermal expansion coefficient, \( \tau_\epsilon \) the thermal relaxation time for temperature gradient (heat flux), \( \tau_{\dot{\epsilon}} \) the thermal retardation time for temperature gradient (heat flux), \( T_i \) the initial temperature at which the cylinder is stress-free, and \( \tau_3 \) the time
parameter that describes the effect of acceleration of strain on temperature. The comma denotes partial differentiation with respect to a variable, and a repeated subscript implies summation. The dot denotes differentiation with respect to time.

For plane strain model with radially symmetric deformation, the only nonzero component of the displacement is the radial displacement component \( u \). The two nonzero strain components are

\[
e_{rr} = \ddot{u} / \ddot{r} \quad e_{\theta\theta} = u / r \quad (2)
\]

The temperature is assumed to be symmetrical about the axis and independent of the axial coordinate \( z \). The material parameters \( k, \rho, c \) and \( e_{\theta} \) are functions of the radial position \( r \).

Then, the heat conduction equation (1) is reduced to

\[
\left( 1 + \tau_1 \right) \frac{\partial}{\partial t} \frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \Theta}{\partial r} \right) = \frac{K}{k} \left( \Theta + \tau_1 \Theta \right) \frac{3 \lambda + 2 \mu}{k} \alpha T_0 \left( \dot{e}_{\theta\theta} + \dot{e}_{\theta r} \right)
\]

(3)

where \( e_{\theta\theta} = e_{\theta\theta} + e_{\theta r} \) is \( \dot{c} / \dot{r} / u + r / r = (1 / r) \dot{u} / \dot{r} \).

The equation of motion for an isotropic FGM without body forces is given by:

\[
\frac{\partial^2 \sigma_{rr}}{\partial r^2} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \ddot{\sigma}_{rr}
\]

(4)

The stress-strain relations for an isotropic FGM is given by

\[
\sigma_{rr} = \lambda (e_{\theta\theta} + 2e_{rr}) \frac{\partial r}{\partial r} + \frac{\lambda c}{r} \left( 3 \lambda + 2 \mu \alpha \right) T_0 \left( T_{rr} + r \dot{r} \right)
\]

(5)

where \( \sigma_{rr} \) is the stress tensor, \( \tau_1 \) the time parameter that describes the effect of temperature gradient on displacement and stress, and \( \delta_{rr} \) the Kronecker’s delta.

Combining (2) and (5) yields the following stress-displacement relations:

\[
\sigma_{rr} = (\lambda + 2 \mu) \frac{\partial u}{\partial r} + \frac{\lambda c}{r} \left( 3 \lambda + 2 \mu \alpha \right) T_0 \left( T_{rr} + r \dot{r} \right)
\]

(6)

The Von Mises stress in the FGM cylinder based on the plane strain assumption is given by

\[
\sigma_{VM} = \sqrt{\sigma_{rr}^2 + \sigma_{\theta\theta}^2 + \sigma_{rr} \sigma_{\theta\theta} - \sigma_{rr} \sigma_{\theta\theta} - \sigma_{rr} \sigma_{rr} - \sigma_{rr} \sigma_{\theta\theta}}
\]

(7)

The material parameters \( \lambda, \mu \), and \( \alpha \) are functions of the radial position \( r \). Substituting (6) into (4) leads to the following equation of motion

\[
\left( \lambda + 2 \mu \right) \frac{d}{dr} \left( \frac{\partial \sigma_{rr}}{\partial r} \right) + \frac{\partial}{\partial r} \left( \frac{\partial \sigma_{rr}}{\partial r} \right) = \rho \ddot{\sigma}_{rr}
\]

(8)

When \( r_1 = r_2 = 0 \), the generalized thermoelastic model is reduced to Lord-Shulman (LS) theory. When \( r_1 = 0 \), and \( r_3 = r_0 \), the generalized thermoelastic model is reduced to the extended LS theory. When \( r_2 = r_3 = 0 \), the generalized thermoelastic model is reduced to Green-Lindsay (GL) theory in which \( r_1 \geq r_0 \geq 0 \). When \( r_0 = r_1 = r_2 = r_3 = 0 \), the generalized thermoelastic model is reduced to the classical linear dynamic theory of thermoelasticity.

### III. MATERIAL PROPERTIES

The FGM compose of two different materials gradually spatially varying. One material is usually ceramic to resist the extreme thermal loading from the high-temperature environment due to its low thermal conductivity. The other material is a metal alloy to maintain the rigidity of the structure and to prevent damage due to thermal stresses initiated by high temperature gradient in a very short period of time.

For a composite material consists of two phases, the effective value of a material property \( P \) of the composite is computed based on the rule of mixtures by:

\[
p = p_m V_m + p_c V_c
\]

(9)

where \( V_m \) and \( V_c \) are the volume fractions of the metal and ceramic, respectively. The volume fractions satisfy the relation \( V_m + V_c = 1 \). The volume fraction of the metal \( V_m \) using the power law is defined as

\[
V_m = V_{m0} + \left( V_{m0} - V_{m1} \right) \left( \frac{r}{r_o - r_b} \right)^n
\]

(10)

where \( r_1 \) is the inner radius of the cylinder, \( r_o \) is the outer radius of the cylinder, \( V_{m0} \) and \( V_{m1} \) are the volume fractions of the metal constituent on the inner and outer surfaces, respectively, and \( n \) is the power law exponent that represents the graded distribution along the radial direction. The cylinder has a linear variation for \( n = 1 \). As the parameter \( n \) increases, the cylinder becomes rich in ceramic when \( V_{m0} = 0 \) and becomes rich in metal when \( V_{m1} = 1 \). Conversely, as \( n \) decreases, the cylinder becomes rich in metal when \( V_{m1} = 0 \) and becomes rich in ceramic when \( V_{m0} = 1 \).

The FGM cylinder considered in this paper is made of Zirconia (ZrO2) and Titanium (Ti-6AL-4V) with negligible porosity. The mechanical and thermal properties of these materials at room temperature are shown in Table I [10]. The material properties are assumed to be independent of temperature.

The properties of FGM are calculated by substituting the volume fractions of the material constituents into the rule of mixtures (9). These properties are expressed as follows:

\[
k(r) = k_m \left( \frac{1 - k_m}{V_m} V_m + \frac{k_m}{k_m} \right)
\]

(11)

\[
\rho(r) = \rho_m \left( 1 - \frac{\rho_m}{\rho_m} V_m + \frac{\rho_m}{\rho_m} \right)
\]

(12)

\[
c_1(r) = c_{m1} \left( 1 - \frac{c_{m1}}{c_{m1}} V_m + \frac{c_{m1}}{c_{m1}} \right)
\]

(13)

\[
E(r) = E_m \left( 1 - \frac{E_m}{E_m} V_m + \frac{E_m}{E_m} \right)
\]

(14)
Lame’s constants are function of Young’s modulus, $E$ and Poisson’s ratio, $\nu$, as follows:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$3\lambda + 2\mu = \frac{E(1-\nu)}{1+\nu}$$

$$\lambda' = \frac{\lambda}{E}\frac{\mu}{\mu}$$

$$\mu' = \frac{\mu}{E}\frac{\lambda}{\lambda}$$

$$\alpha = \sqrt{\frac{\lambda}{\mu}}$$

**Table I**

<table>
<thead>
<tr>
<th>Material</th>
<th>$k_c$ (W/mK)</th>
<th>$E_c$ (GPa)</th>
<th>$\alpha$</th>
<th>$\nu$</th>
<th>$p_c$ (kg/m$^3$)</th>
<th>$c_v$ (J/(kgK))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ti-6AL-4V</td>
<td>18.1</td>
<td>66.2</td>
<td>10.3</td>
<td>0.321</td>
<td>4.42</td>
<td>808.3</td>
</tr>
<tr>
<td>ZrO$_2$</td>
<td>2.036</td>
<td>117.0</td>
<td>7.11</td>
<td>0.333</td>
<td>5.6</td>
<td>615.6</td>
</tr>
</tbody>
</table>

**IV. MATHEMATICAL MODEL**

Initially, the temperature of the cylinder is assumed to be uniform and equal to the ambient temperature, $T_a$. To represent a thermal shock, temperature at the inner surface of the cylinder is suddenly elevated to a new value, $T_i$, while the outer surface temperature, $T_o$ is maintained at the reference temperature, $T_a$. Therefore, the initial and boundary conditions for the thermal field are written as

$$T(r,0) = T_o, \quad \frac{\partial T}{\partial r}(r,0) = 0, \quad T(r,t) = T_i, \quad T(r,t) = T_a$$

(18)

The thermal stresses are free, $\sigma_0 = 0$, at the initial temperature $T_o$. The traction free boundary conditions at the inner and outer surfaces of the cylinder are assumed. Therefore, the initial and boundary conditions for the thermoelastic field are written as

$$u(r,0) = 0, \quad \frac{\partial u}{\partial r}(r,0) = 0$$

$$\sigma(r,t) = 0, \quad \sigma_0(r,t) = 0$$

(19)

The governing equations can be written in a more suitable form by using the following dimensionless variables

$$\theta = \frac{T - T_o}{T_i - T_o}, \quad R = \frac{r}{r_i}, \quad \zeta = \frac{t}{t_s}, \quad t_s = \frac{r_o c_v p_o}{\alpha T_{io}^2} r_i^2$$

$$\beta_i = \frac{T_i}{T_o} (i = 0,1,2,3)$$

$$\kappa' = \frac{k}{k_o}, \quad \rho' = \frac{\rho}{\rho_o}, \quad c_i' = \frac{c_i}{c_o}$$

$$E' = \frac{E}{E_o}, \quad \alpha' = \frac{\alpha}{\alpha_o}, \quad U = \frac{U}{r_o a(T_i - T_o)}, \quad R^* = \frac{r}{r_i}$$

$$\lambda' = \frac{\lambda}{E}\frac{\mu}{\mu}$$

$$\mu' = \frac{\mu}{E}\frac{\lambda}{\lambda}$$

$$\alpha = \sqrt{\frac{\lambda}{\mu}}$$

$$S_i = \frac{\sigma_i}{E_o a(T_i - T_o)} (i = r, \phi, z)$$

(20)

where $\delta$ represents the non-dimensional thermoelastic coupling constant.

By using these dimensionless variables, the following non-dimensional governing equations obtained

$$\frac{1 + \beta_i \theta}{c_i} \frac{\partial \theta}{\partial t} + \frac{\alpha' \mu'}{a(t) \alpha' \alpha'} \frac{\partial \theta}{\partial \theta} + \frac{\beta' \rho'}{\rho_0 a} \frac{\partial \theta}{\partial \rho} - \frac{\kappa' T}{c_i} \frac{\partial \theta}{\partial \lambda} = 0$$

(21)

$$\frac{\delta}{\delta U} \left( \theta + \beta \frac{\partial \theta}{\partial \delta} \right) - \frac{\beta \theta}{\delta R} \frac{\partial \theta}{\partial \delta} = 0$$

(22)

Then the dimensionless initial and boundary conditions are given by:

$$\theta(R,0) = U(R,0) = 0$$

$$\theta(\delta,0) = \theta(0,\delta) = 0$$

(23)

$$\theta(R,\zeta) = \theta(\delta,\zeta) = 0$$

$$S_i(\delta,\zeta) = S_i(0,\zeta)$$

**V. NUMERICAL RESULTS AND DISCUSSION**

The FGM hollow cylinder considered in this paper has an outer radius of $r_o = 150$ mm and inner radius of $r_i = 50$ mm. The inner surface temperature is suddenly increased to $T_i = 1800$ K and then kept constant, while the outer surface temperature is maintained at the reference temperature, that is, $T_o = 300$ K. Also, the cylinder is assumed to be traction free at the inner and outer surfaces. The generalized coupled thermoelasticity based on the Green-Lindsay (GL) model is considered in this paper.

When $r_2 = r_3 = 0$, the generalized thermoelastic model is reduced to Green-Lindsay (GL) theory in which $r_2 \geq r_3 \geq 0$. The numerical values of the relaxation times $(t_s = t_{so})$ are assumed to be $10-100$ s.

The solution of the heat conduction equation (20) and the equation of motion (21) are obtained by implementing the Galerkin finite element method. All the finite element calculations were done with a commercial finite element package FlexPDE.
Figures 1-5 show the variations at different times of the temperature, radial displacement, radial stress, hoop stress and axial stress, respectively, along the radial direction of the cylinder, for the case of $V_{m,j} = 0$, $V_{m,r} = 1$ and $n = 1$. This means that the inner and outer surfaces of the cylinder are fully ceramic and fully metal, respectively. The axial and hoop stresses are compressive at the inner surface of the cylinder and are tensile at the outer surface as shown in Figs. 4-5. The compressive stress has a maximum value on the inner surface at the beginning of the heating process and then decreases suddenly with distance from the inner surface. As shown from the figures, the extreme stress gradients occur at the heated inner surface. The thermal stress distributions approach the steady state distribution as the time increases.
VI. CONCLUSION

The dynamic thermoelastic response of a hollow thick cylinder made of functionally graded material subjected to thermal loading is investigated. The generalized coupled thermoelasticity theory based on the Green-Lindsay model is considered. The material composition of the cylinder is graded through the radial direction of the cylinder. The thermal and mechanical properties of the FGM cylinder are obtained by using the rule of mixtures scheme, where a power law distribution is assumed for the volume fraction of the material constituents.

The FG cylinder is assumed to be symmetrically loaded and one-dimensional transient analysis of isotropic linear thermoelastic FG cylinder under thermal loading is investigated. The heat conduction equation and the equation of motion are solved numerically by using the Galerkin finite element method. The transient temperature, radial displacement, radial stress, hoop stress and axial stress distributions through the radial direction of the cylinder are shown.

REFERENCES