A $C^1$-Conforming Finite Element Method for Nonlinear Fourth-Order hyperbolic Equation

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Abstract—In this paper, the $C^1$-conforming finite element method is analyzed for a class of nonlinear fourth-order hyperbolic partial differential equation. Some a priori bounds are derived using Lyapunov functional, and existence, uniqueness and regularity for the weak solutions are proved. Optimal error estimates are derived for both semidiscrete and fully discrete schemes.

Keywords—Nonlinear fourth-order hyperbolic equation; Lyapunov functional; Existence, uniqueness and regularity; Conforming finite element method; Optimal error estimates.

I. INTRODUCTION

In this paper, the $C^1$-conforming finite element method is analyzed for the following fourth-order hyperbolic equation

$$u_{tt} + \gamma \Delta^2 u - \Delta u - \Delta u_t + f(u) = 0, \ (x, t) \in \Omega \times J,$$

with boundary conditions

$$u(x, t) = 0, \ \frac{\partial u}{\partial n} = 0, \ (x, t) \in \partial \Omega \times J,$$

or

$$u(x, t) = 0, \ \Delta u = 0, \ (x, t) \in \partial \Omega \times J,$$

and initial condition

$$u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ x \in \Omega,$$

where $f(u) = u^3 - u$ and $\Omega$ is a bounded domain in $\mathbb{R}^d$, $d \leq 2$ with boundary $\partial \Omega$, $J = (0, T]$ with $0 < T < \infty$.

In recent years, much attention has been given in the literature to the numerical solution of fourth-order partial differential equations. In [1], the finite element method was studied for the fourth-order Cahn-Hilliard equation. In [2], the stabilized finite element approximation for fourth order obstacle problem was discussed. In [3], the approximation for the fourth-order eigenvalue problem with cubic Hermite elements on anisotropic meshes was investigated. Chen [4] proposed the expanded mixed finite element method for fourth-order parabolic elliptic equations. In [5], [6], the mixed time discontinuous finite element method was proposed for fourth-order parabolic partial differential equations. Li [7], [8], [9], [10], discussed some mixed finite element methods for fourth-order elliptic problems and parabolic problems. In [11], a new nonconforming element constructed by the Double Set Parameter method, is applied to the fourth order elliptic singular perturbation problem. In [12], the $C^1$-conforming finite element method is analyzed for the extended Fisher-Kolmogorov (EFK) equation, and some numerical results are given to illustrate the efficiency of the method.

In this paper, our purpose is to discuss and analyze the $C^1$-conforming finite element method [12] for the fourth-order hyperbolic equation. We derive some a priori bounds by using Lyapunov functional, and prove the existence, uniqueness and regularity for weak solutions, and obtain the optimal error estimates for both the semidiscrete and fully discrete schemes. Throughout this paper, $C$ will denote a generic positive constant which does not depend on the spatial mesh parameter $h$ and time discretization parameter $\Delta t$.

II. EXISTENCE, UNIQUENESS AND REGULARITY FOR WEAK SOLUTIONS

In this section, we give existence uniqueness and regularity results for the fourth-order hyperbolic equation.

Take $L^2$-inner product of (1) with $v \in H_0^1$ and apply Green’s formula to obtain the following weak formulation

$$\int_{\Omega} (u_{tt} + \gamma \Delta^2 u + \Delta u_t + f(u)) v \, dx + \int_{\Omega} \Delta u \frac{\partial u}{\partial n} v \, dS = 0$$

(5)

with $u(0) = u_0, u_t(0) = u_1$.

To prove existence and uniqueness results, we will derive a priori bound.

Lemma 2.1: Assume that $u_0 \in H_0^1$ and $u_1 \in H^1$. Then there exists a positive constant $C$ such that

$$||u(t)||_2 + ||u(t)||_\infty + ||u_t||_1 \leq C(\gamma_1 ||u_0||_2, ||u_1||_1), \ t > 0.$$

Proof. We consider the Lyapunov functional $\Xi(v)$ as

$$\Xi(v) = \int_{\Omega} \frac{1}{2} |\Delta v|^2 + \frac{1}{2} |\nabla v|^2 + \frac{1}{2} |\nabla u_t|^2 + F(v) \, dx,$$

(6)

where

$$F(v) = \frac{1}{4} (1 - v^2)^2.$$

Note that $F' = f$. Differentiating (6) with respect to $t$ and using (5), we obtain

$$\frac{d}{dt} \Xi(u) = \gamma (\Delta u_t, \Delta u_t) + (\nabla u, \nabla u_t) + (\nabla u_t, \nabla u_{tt}) + (f(u), u_t) = -\frac{d}{dt} ||u_t||^2.$$

(7)

Integrating (7) from 0 to $t$, we obtain

$$\Xi(u) = \Xi(u(0)) + 2 ||u_t(0)||^2 - 2 ||u_t||^2 \leq \Xi(u(0)) + 2 ||u_t(0)||^2,$$

$$||u_t||^2 = \frac{1}{2} \Xi(u(0)) + ||u_t(0)||^2 - \frac{1}{2} \Xi(u) \leq \frac{1}{2} \Xi(u(0)) + ||u_t(0)||^2.$$

(8)
Since $F(u) \geq 0$, using Poincaré inequality, we obtain
\[ ||v||_2 + ||v_t||_1 \leq C(\gamma, ||u_0||_2, ||u_1||_1). \] (9)

Apply the Sobolev imbedding theorem to obtain
\[ ||u||_{L^\infty} \leq ||u||_{H^2} \leq C(\gamma, ||u_0||_2, ||u_1||_1). \] (10)

In the following subsequent sections, we will discuss the global existence, uniqueness and regularity.

**Theorem 2.2:** Let $u_0 \in H^2_0(\Omega)$, $u_t \in H^1(\Omega)$. There exists a unique $u = u(x, t) \in \Omega \times (0, T]$ with
\[ u \in L^\infty(J; H^2_0(\Omega)), \ u_t \in L^\infty(J; H^1(\Omega)). \]
such that $u$ satisfies the initial condition $u(0) = u_0, u_t(0) = u_1$ and the equation (5) in the sense that
\[ (u_{tt}, v) + \gamma(\Delta u, \Delta v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) + (f(u) - f(u), v) = 0, \ v \in H^2_0(\Omega), \ t \in (0, T]. \] (11)

**Proof.** Use the similar method to [12] to obtain the existence of equation (5). Below, we will prove the uniqueness of weak solutions.

Suppose $u$ and $w$ are two solutions of (5). Taking $\sigma = u - w,$ we can obtain
\[ (\sigma_{tt}, v) + \gamma(\Delta \sigma, \Delta v) + (\nabla \sigma, \nabla v) + (\nabla \sigma_t, \nabla v) + (f(u) - f(w), v) = 0, \ v \in H^2_0(\Omega). \] (12)

Setting $v = \sigma_t$ and using the boundedness of $||u||_{L^\infty}$ and $||u||_{L^\infty}$, we have
\[ \frac{1}{2} d \frac{dt}{dt} (||\sigma_t||^2 + 2 + ||\nabla \sigma_t||^2 + ||\nabla \sigma||^2) \leq -(f(u) - f(w), \sigma_t) = -(f'(\xi)\sigma_t, \sigma_t) \leq C\int_0^t ||\sigma_t(s)||^2 ds + ||\sigma_t||^2. \] (13)

Integrating (13) from 0 to $t$, we obtain
\[ ||\sigma_t||^2 + 2 + ||\nabla \sigma_t||^2 + \int_0^t ||\nabla \sigma_t||^2 ds \leq C\int_0^t ||\sigma_t(s)||^2 ds. \] (14)

Using the Gronwall Lemma, we have $||\sigma_t||^2 = 0$ and $\int_0^t ||\sigma_t(s)||^2 ds = 0$, which imply $\sigma_t = 0$.

Note that $\sigma(0) = u(0) - w(0) = 0$ to have $\sigma = \int_0^t \sigma_t dt = 0$.

**III. ERROR ESTIMATES FOR SEMIDISCRETE SCHEMES**

In this section, we apply Galerkin procedure for the fourth-order hyperbolic equation and obtain the semidiscrete scheme and a priori error estimates.

Let $S_h^0$, $0 < h < 1$ be a family of finite dimensional subspace of $H^2_0$ with the following approximation property ([14]): For $v \in H^1(\Omega) \cap H^2(\Omega)$, there exists a constant $C$ independent of $h$ such that
\[ \inf_{\chi \in S_h^0} ||v - \chi||_j \leq Ch^{j-1}||v||_4, \ j = 0, 1, 2. \] (15)

The corresponding semidiscrete Galerkin approximation of (1)-(4) is defined to be a function $u_h : [0, T] \to S_h^0$ such that
\[ \frac{1}{2} \int_0^T ||\xi_t||^2 d\tau + \int_0^T ||\xi||^2 d\tau = (\eta(0), \xi) + \int_0^T \eta(\xi_t - f(u) + f(u) d\tau. \] (16)

To obtain optimal rate of convergence, following wheeler[13], we introduce $\tilde{u}$ be as an auxiliary projection of $u$ defined by
\[ A(u - \tilde{u}, v_h) = 0, \ v_h \in S_h^0, \] (17)
where the bilinear form is introduced by
\[ A(v, w) = \gamma(\Delta v, \Delta w) + (\nabla v, \nabla w) + (\nabla \xi_t, \nabla \xi), v, w \in H^2_0, \]
for our subsequent use note that $A(\cdot, \cdot)$ satisfies the following properties:
(i) Boundedness: There is a positive constant $M$ such that
\[ ||A(v, w)|| \leq M||v||_2||w||_2, v, w \in H^2_0. \]
(ii) Coercivity: There is a constant $a_0 > 0$ such that
\[ A(v, v) \geq a_0||v||_2^2, v \in H^2_0. \]

With $\eta = u - \tilde{u}$, the following estimates are well known [12], [13]: for $j = 0, 1, 2$
\[ ||\eta||_{L^\infty} \leq Ch^{j-1} \sum_{k=0}^j ||\eta||_{L^2_0}. \] (18)

Assuming quasi-uniformity condition on the triangulation, it is easy to check that
\[ ||\eta(t)||_{L^\infty} \leq Ch^{j-1}||u(t)||_{L^4}, \ j = 0, 1. \] (19)

For a priori error estimates, we decompose the errors as
\[ u - u^h = u - \tilde{u} + \tilde{u} - u^h = \eta + \xi. \]

**Theorem 3.1:** There exists a positive constant $C$ independent of $h$ such that
\[ ||u_t - u_{ht}||_{L^2(0, T, L^2(\Omega))} \leq Ch^{j-1}(||u_t||_{L^2(0, T, L^2(\Omega))} + ||u||_{L^\infty(0, T, L^2(\Omega))}). \]

Moreover, assuming quasi-uniformity condition on the triangulation $T_h$, there exists a positive constant $C$ independent of $h$ such that
\[ ||u - u_h||_{L^\infty(0, T, L^2(\Omega))} \leq Ch^{j}(||u||_{L^\infty(0, T, L^2(\Omega))} + ||u_t||_{L^2(0, T, L^2(\Omega))}). \]

**Proof.** Subtracting (16) from (5) and using auxiliary projection, we obtain the following equation in $\xi$
\[ (\xi_t, v_h) + \gamma(\xi_t, \xi) + (\nabla \xi_t, \nabla v_h) + (\nabla \xi_t, \nabla v_h) \]
\[ = -((\eta_t, v_h) - (f(u) - f(u), v_h). \] (20)
Choosing $v_h = \xi_t$ in (20), and using the Cauchy-Schwarz inequality and the Poincaré’s inequality, we have
\[ \frac{1}{2} \int_0^T ||\xi_t||^2 d\tau + \gamma(\xi_t, \xi) + ||\nabla \xi_t||^2 \leq C(||\eta||^2 + ||\eta_t||^2 + ||\xi||^2 + ||\nabla \xi||^2) \]
\[ \leq C(||\eta||^2 + ||\eta_t||^2 + ||\xi||^2 + ||\nabla \xi||^2). \] (21)
Integrating (21) from 0 to t, we obtain

\[ \frac{d}{dt} \left( t \frac{dU}{dt} \right) + \frac{\gamma}{2} \| \Delta U \|^2 + \frac{1}{2} \| \nabla U \|^2 \leq C \int_0^t \left( \frac{d}{ds} \| \Delta U \|^2 + \frac{\gamma}{2} \| \Delta U \|^2 + \frac{1}{2} \| \nabla U \|^2 \right) ds \]

Using the Gronwall’s lemma, we have

\[ \| \Delta U \|^2 + \frac{1}{2} \| \nabla U \|^2 \leq C \int_0^t \left( \frac{d}{ds} \| \Delta U \|^2 + \frac{\gamma}{2} \| \Delta U \|^2 + \frac{1}{2} \| \nabla U \|^2 \right) ds \]

Substituting the estimates of \( \| \Delta U \| \) and \( \| \nabla U \| \) and using the Poincaré’s inequality, we obtain the following superconvergence result for \( \| \nabla U \| \)

\[ \| \Delta U \|^2 + \frac{1}{2} \| \nabla U \|^2 \leq C \| \Delta U \|^2 \]

and hence,

\[ \| \Delta U \|^2 + \frac{1}{2} \| \nabla U \|^2 \leq C \| \Delta U \|^2 \]

We apply the triangle inequality to get the the conclusion.

IV. Fully discrete schemes and error estimates

In this Section, we briefly describe a fully discrete scheme for approximating the solution of (1) and discuss a priori error bounds.

Let \( 0 = t_0 < t_1 < t_2 < \ldots < t_M = T \) be a given partition of the time interval \([0, T]\) with step length \( \Delta t = T / M \), for some positive integer \( M \). We use the following notation related to functions defined at discrete time levels. For a smooth function \( \phi \) on \([0, T]\), define

\[ \phi^n = \phi(t_n), \quad \phi^{n+\frac{1}{2}} = \frac{1}{2} (\phi^{n+1} + \phi^n), \quad \phi_t^n = \frac{\phi^{n+\frac{1}{2}} - \phi^{n-\frac{1}{2}}}{\Delta t}, \quad \phi_{tt}^n = \frac{\phi^{n+\frac{1}{2}} - 2\phi^n + \phi^{n-\frac{1}{2}}}{\Delta t}, \quad \phi_t^{n+\frac{1}{2}} = \frac{1}{2} (\phi^{n+\frac{3}{2}} + \phi^{n-\frac{1}{2}}), \]

\[ G(\chi, \psi) = \frac{F(\chi) - F(\psi)}{\chi - \psi} \]

Let \( U^n \) be the approximations of \( u \) at \( t = t_n \) which we shall define through the following scheme. Given \( U^{n-1} \) in \( V_h \), we now determine a pair \( U^n \) in \( V_h \) satisfying

\[ \begin{align*}
&\left( \frac{2}{\Delta t} \partial_t U^{n+\frac{1}{2}}, v_h \right) + \gamma (\Delta U^{n+\frac{1}{2}}, \Delta v_h) + (\nabla U^{n+\frac{1}{2}}, \nabla v_h) + (\nabla \partial_t U^{n+\frac{1}{2}}, \nabla v_h) + (G(U^{1}, U^n), v_h) = \left( \frac{2}{\Delta t} \partial_t U^{n+\frac{1}{2}}, v_h \right) \quad \forall \ v_h \in S_h^n, \\
&\left( \frac{2}{\Delta t} \partial_t U^{n+\frac{1}{2}}, v_h \right) + \gamma (\Delta U^{n+\frac{1}{2}}, \Delta v_h) + (\nabla U^{n+\frac{1}{2}}, \nabla v_h) + (G(U^{1}, U^n), v_h) = 0, \quad v_h \in S_h^n, \ n \geq 1, \\
\end{align*} \]
For fully discrete error estimates, we now split the errors \( u(t_n) - U^n = (u(t_n) - \bar{u}(t_n)) + (\bar{u}(t_n) - U^n) = \eta^n + \xi^n \)

Use (5), (17), and (27), then we obtain

\[
\begin{align*}
\left( \frac{\Delta t}{2} \partial_t \xi_n^1 + v_n \right) + \gamma (\Delta \xi_n^1, \Delta v_n) + (\nabla \xi_n^1, \nabla v_n) + (\nabla \partial_t \xi_n^1, \nabla v_n) + (f(u(t_n)) - G(U^n, U^{n-1}), v_n) \leq C \left[ ||\partial_t \xi_n^1||^2 + ||\Delta \xi_n^1||^2 + ||\nabla \xi_n^1||^2 + ||\nabla \partial_t \xi_n^1||^2 \right],
\end{align*}
\]

Using the Gronwall lemma and (41) to obtain

\[
||\partial_t \xi_n^1||^2 + ||\Delta \xi_n^1||^2 + ||\nabla \xi_n^1||^2 + ||\nabla \partial_t \xi_n^1||^2 \leq C \left[ ||\partial_t \xi_n^1||^2 + ||\Delta \xi_n^1||^2 + ||\nabla \xi_n^1||^2 + ||\nabla \partial_t \xi_n^1||^2 \right] + C \Delta t^4.
\]

For the estimation of the first three terms on the right-hand side of the above inequality, we now choose \( w_n = \partial_t \xi_n^1 \) in (36) to obtain

\[
\begin{align*}
\left( \frac{\Delta t}{2} \partial_t \xi_n^1, \partial_t \xi_n^1 \right) + (\Delta \xi_n^1, \Delta \xi_n^1) + (\nabla \xi_n^1, \nabla \xi_n^1) + (\nabla \partial_t \xi_n^1, \nabla \xi_n^1) + (f(u(t_n)) - G(U^n, U^{n-1}), \partial_t \xi_n^1) \leq C \Delta t^4 + ||\partial_t \xi_n^1||^2.
\end{align*}
\]

Noting that

\[
\begin{align*}
||\partial_t \xi_n^1||^2 + ||\partial_t \xi_n^1\xi_n^1||^2 \leq \frac{1}{2 \Delta t} ||\partial_t \xi_n^1||^2 + ||\partial_t \xi_n^1\xi_n^1||^2, \\
||\Delta \xi_n^1||^2 + ||\Delta \xi_n^1\xi_n^1||^2 \leq \frac{1}{2 \Delta t} ||\Delta \xi_n^1||^2 + ||\Delta \xi_n^1\xi_n^1||^2, \\
||\nabla \xi_n^1||^2 + ||\nabla \xi_n^1\xi_n^1||^2 \leq \frac{1}{2 \Delta t} ||\nabla \xi_n^1||^2 + ||\nabla \xi_n^1\xi_n^1||^2, \\
||f(u(t_n)) - G(U^n, U^{n-1})|| \leq \frac{1}{\Delta t} ||f(u(t_n)) - G(u(t_n), u(t_{n-1}))|| + ||G(u(t_n), u(t_{n-1})) - G(U^n, U^{n-1})|| \leq C \Delta t^2.
\end{align*}
\]
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