Portfolio Management: A Fuzzy Set Based Approach to Monitoring Size to Maximize Return and Minimize Risk

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Abstract—Fuzzy logic can be used when knowledge is incomplete or when ambiguity of data exists. The purpose of this paper is to propose a proactive fuzzy set-based model for reacting to the risk inherent in investment activities relative to a complete view of portfolio management. Fuzzy rules are given where, depending on the antecedents, the portfolio size may be slightly or significantly decreased or increased. The decision maker considers acceptable bounds on the proportion of acceptable risk and return. The Fuzzy Controller model allows learning to be achieved as 1) the firing strength of each rule is measured, 2) fuzzy output allows rules to be updated, and 3) new actions are recommended as the system continues to loop. An extension is given to the fuzzy controller that evaluates potential financial loss before adjusting the portfolio. An application is presented that illustrates the algorithm and extension developed in the paper.

Keywords—Portfolio Management, Financial Market Monitoring, Fuzzy Controller, Fuzzy Logic,

INTRODUCTION

RISK assessment models and strategies to control risks suggested in the literature do not take into account ambiguity. In recent models authors have relied on probability theory to present their solution to the problem of risk assessment. However, it has been long established that probability models do not address the problem of ambiguity [1].

Ellsberg [2] was one of the earliest authors to demonstrate that the fundamental rules governing probability are rendered invalid if the likelihood of events cannot be expressed in binary terms. A number of subsequent empirical studies (for example, [3], [4], [5], [6]) have confirmed Ellsberg’s paradox.

Since the assessment of risks in the course of portfolio selection cannot be expressed in quantifiable terms, experts have to often rely on verbal expressions. Such verbal expressions entail both uncertainty and ambiguity. Uncertainty is caused by the fact that the threats are prospective in nature and the possibility of their occurrence cannot be foreseen in precise terms. Ambiguity in responses to questions asked is attributable to the lack of precision in verbal descriptions, such as “high” ‘moderate” or “low”. In view of the fact that traditional two-valued logic of probability judgments is inadequate to handle the combined presence of uncertainty and ambiguity, the calculus of fuzzy sets is utilized in the proposed model to allow effective and systematic handling of imprecise and imperfect information.

A fuzzy set controller is discussed in Section III after presentation of the background on financial risk assessment in portfolio management with its primary reliance on non-fuzzy set based techniques and then a description of fuzzy logic fundamentals including fuzzy controllers in Section II. Application is made to an investment situation within the typical limits of 99% in Section IV. Section V presents conclusions and benefits of this fuzzy controller approach.

II. BACKGROUND

A selection process for a stock portfolio was developed by Markowitz ([7], [8]). The seminal work by Markowitz [7] proposes a vector $x$ of asset weights which sum to 1 be chosen such that linear combinations $\mu[\text{sub } p] = x[\sup '][\mu$ of the expected asset returns $\mu$ (expected excess returns) which represent the expected return on a portfolio, maximized for a specified level of “risk”, $\sigma[\text{sub } p]$, (the standard deviation of the portfolio). The efficient frontier is the curve of $(\mu[\text{sub } p], \sigma[\text{sub } p])$ traced by portfolios whose return/risk tradeoff is optimal in this sense [9].

Continuous time, mean-variance portfolio selection has been studied using a stochastic linear-quadratic (LQ) optimal control and backward stochastic differential equations (BSDEs). The LQ problem was reduced to solving a stochastic Riccati equation which was a fully nonlinear and singular BSDE with random coefficients. It was demonstrated that the efficient frontier in the mean-standard deviation diagram was a straight line and as such risk-free investment is a distinct possibility even when the interest rate is random [10].

A. Basics of Financial Risk:

As early as 1971, the estimation of risk in portfolio selection models was recognized to be problematic since the true values of the parameters of the distribution of returns are

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chooses a portfolio solely on expected value and variance (marginal quantiles are known and uncertainty is evident. be performed quickly. The lower bound is thus interpreted as computational ease in situations where risk evaluation has to approach was proposed by Luciano and Marena [17] for its extensively. The optimal portfolio selection problem under Knightian uncertainty considers the decision maker's portfolio consisting of one risky and one risk-free asset. Expected utility is used to derive bounds on the no-transaction region for both optimistic and pessimistic decision makers resulting in a closed interval for the standard expected utility but not necessary closed for the Choquet expected utility [18]. Thus, types of standard utility functions can impact the transaction area in both risk and risk-free scenarios.

Sharpe [19] was one of the first to propose a simplified method for the efficient set of portfolios under the assumptions of regression theory. The maximum Sharpe ratio determines the portfolio with the highest return/risk tradeoff achievable from the assets. Recently, therefore, an approach estimating mean excess returns and their covariance matrix has been developed to determine the maximum Sharpe matrix [9].

Since market and credit risks are generally modeled simultaneously, separate risk factors associated with pure market portfolio or pure credit portfolio models, have been tested by both linear and non-linear models [20]. Linear programming has been used to simultaneously minimize general risk or dispersion measures by first transforming the general risk minimization problem to a minimax problem, setting up necessary and sufficient conditions and solving the dual problem for both optimal portfolios and their sensitivities [21]. A minimax type risk function was also studied by Teo and Yang [22] in which the investor attempts to restrict the standard deviation for each of the available stocks. Again, the portfolio optimization problem is formulated as a linear programming problem based on capital asset pricing between the market portfolio and each individual return using nonsmooth methods. A conic programming has been used in a worst case Value-at-Risk (VaR) and robust portfolio optimization problem [16].

The above examples relate exclusively to the problems inherent in uncertainty in portfolio management yet each uses stochastic approaches, simulation or linear programming/regression to address the problems. Indeed, uncertainty utilizing fuzzy sets is underrepresented in the literature on portfolio management. Duval and Featherstone [23] developed a fuzzy logic alternative to the traditional mean variance model and compromise programming approach. The model's application was focused on the agribusiness industry. Thus, while it had results that suggested investments in publically traded food and agribusiness stocks allow farmers to capture additional benefits beyond market diversification options, its focus was limited. Importantly, given the previous reliance on stochastic approaches, minimax models, linear programming, and/or simulation models, the use of fuzzy logic provided realistic results with an innovative approach to address uncertainty in decision making.

More in line with the focus of portfolio management, a fuzzy logic controller was posited by Khoshnevisan et al. [24] to minimize cumulative hedging error based on the Black-Scholes [25] options pricing functional form. This model formulated a risk-free scenario such that the minimum return was equal to the risk-free rate r and expected present value of the terminal option payoff seeks the maximum of the wealth invested or the expected return on the risk-free asset over the investment horizon. The selected tracking error function utilized utility functions where the replicating portfolio value at time t is linear. The model then considers the best performer to be least risky. Based on a simple bGA model, a fuzzy logic
controller (FLC) process was proposed as the ideal interface between man and machine allowing prioritization between exploitation and exploration for long term options. The FLC was utilized such that at each periodic rebalancing point, k percent of funds are allocated to the best-performing risk asset with (90-k) percent to the other risky asset to keep the portfolio self-balancing after the initial investment. A rule system determined that k would be raised or lowered slightly if the square of the risk differential is small. The decision process is thus considered over time and an interpretation of risk.

### B. Basics of Fuzzy Logic for Risk

Fuzzy logic ([26], [27]) can be used to make decisions, where a fuzzy subset A of a set X is a function of X into \([0,1]\). We can think of fuzzy probabilities as:

\[
D = \{x \in C_j, y \in y \}\text{is}\hat{0}\}
\]

where \(A(x) = 0\) for  \(x = a\).

Corresponding fuzzy benefits are defined by fuzzy sets \(B_{ik}\) where \(B_{ik} = \Sigma_{x = b_{ik}} / b_{ik}\).

Then using the extension principle ([27]), the averaged benefit is defined by:

\[
B = \Sigma I_k / b_k \text{ where } \sum I_k = 1
\]

where \(b_k = \min\{I_k\} / a_k\).

If \(\sum a_k = 1\) then:

\[
\sum a_k b_k = \sum I_k / b_k = \max\ a_k b_k
\]

\[
A(x) = 0 \text{ if } x < a
\]

\[
A(x) \geq A(y) \text{ if } a \leq y \leq x \leq b
\]

\[
A(x) = A(y) \text{ if } b \leq y \leq x \leq c
\]

\[
A(x) \leq A(y) \text{ if } c \leq y \leq x \leq d
\]

\[
A(0) = 0 \text{ for } x \geq d
\]

The most common trapezoidal set is the triangular shape where \(b = c\). Then, if \(A\) is a fuzzy set by the \(\alpha\)-level of \(A\) we mean the set:

\[
A(x) = \{x \mid A(x) \geq \alpha\}
\]

where \(\alpha\) is identified with its characteristic function \(A(x) = 1\) if \(A(x) \geq \alpha\) and \(A(x) = 0\) otherwise.

\(A\)  Algorithm

1. Consider the \(P_i\) rule:

\[
R_j: \text{when } V \text{ is } A_j \text{ and } U \text{ is } B_j \text{ then } W \text{ is } C_j
\]

2. Associated to the rule \(R_j\), define a fuzzy set \(F_j\) by:

\[
F_j = A_j \cap B_j \cap C_j
\]

3. Define the measure of how well \(x, y, z\) fits rule \(R_j\) as:

\[
F_j(x, y, z) = \min\{A_j(x), B_j(y), C_j(z)\}
\]

where \(x, y, z\) are in the domains of \(A_j, B_j,\) and \(C_j\).

4. Assume the following:

\[
V = A \text{ and } U = B \text{ where } A \text{ and } B \text{ are fuzzy sets.}
\]

5. Define \(D_i = F_j \cap A \cap B\) where \(D_i\) measures how well the input "V = A" and "U = B" fits rule \(R_j\).

6. Let \(\max D_i\) denote the maximum \(x\) ranges over all of its allowable value, such that \(\max D_i\).
\{A_j(x), A(x)\} denotes the possibility of having \(A_j\) given \(A\). We denote this possibility by \(\text{Poss}[A_j | A]\).

7. Define fuzzy set \(S_j\) for rule \(R_j\) by:
\[
S_j(z) = \text{Minimum} \{\text{Poss}(A_j | A), \text{Poss}(B_j | B), C_j(z)\}
\]
where \(S_j(z)\) measure for \(R_j\) how much \(z\) fits the then part of the rule, given that the inputs are \(U = A\) and \(V = B\).

8. Let \(T_j = \text{Minimum}\{A_j(x_0), B_j(y_0)\}\) where \(x_0, y_0\) is crisp input (given by sensors) Minimum \{Poss (A_j | A), Poss (B_j | B)\} = \(T_j\) if \(A\) and \(B\) are crisp \(x_0\) and \(y_0\).

9. Then \(T_j\) measure how strongly \(R_j\) should apply given that we have inputs \(U = X_0\) and \(V = Y_0\) where \(X_0\) and \(Y_0\) are crisp values.

10. Then, \(S_j = \text{Minimum}\{T_j, C_j(z)\}\) denotes how rule \(R_j\) is fired when the then part of the rule \(C_j(z)\) is dampened by the firing strength \(T_j\) of the rule \(R_j\).

11. Define \(S(z) = \text{Maximum}_j S_j(z)\) as how much action \(z\) should be taken given the input and given the fact that we wish to fire all rules \(R_j\) with strength \(T_j\).

12. Define \(S(z) = \{\sum_j T_j \times S_j(z)\}/\sum_j T_j\) as the weighted average of all information coming from all rules.

13. Consider the \(z_j\) level of each \(S_j\) as: \(C_T_j = [a_j, b_j]\) as an interval on the convex set; \(C_T_j = [a_j, b_j]\).

14. Consider the midpoint \(m_j\) of \(C_T_j\) as: \(m_j = (a_j + b_j)/2\)

15. Defuzzifying for action \(z\) is:
\[
z = [\sum_j m_j T_j]/\sum_j T_j
\]
(This algorithm is due to Yager [36] and Shipley, et al. [37].)

IV. APPLICATION

An example of how a fuzzy controller can be used to monitor and react to variations in a typical portfolio management problem is presented. The application concerns re-evaluation of a portfolio over the previous 20 days (approximately one month of trading opportunity) in which the risk/return ratio has been as high as 14% but in which the average is 4%. The following application is based on Table 1 below.

### TABLE 1.
Risk/Return over 20-Day Personal Investments of 50-Stock Portfolio

<table>
<thead>
<tr>
<th>Day Number</th>
<th>Risk/Return Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.04</td>
</tr>
<tr>
<td>2</td>
<td>.06</td>
</tr>
<tr>
<td>3</td>
<td>.08</td>
</tr>
<tr>
<td>4</td>
<td>.02</td>
</tr>
<tr>
<td>5</td>
<td>.00</td>
</tr>
<tr>
<td>6</td>
<td>.04</td>
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<td>7</td>
<td>.08</td>
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<td>8</td>
<td>.02</td>
</tr>
<tr>
<td>9</td>
<td>.02</td>
</tr>
<tr>
<td>10</td>
<td>.06</td>
</tr>
</tbody>
</table>

Using 99% limits the following chart can be constructed with the risk/return of the portfolio plotted as shown in Figure 2 below.

Generally, the risk/return for one day is out of the acceptable limits of the investor. Since this could significantly impact the financial status of the investor depending on the amount of the trade, the controller could be used to make adjustments to reduce the portfolio size. On the other hand, if all risk/return proportions are clustered close to the mean, the controller would suggest appropriate reduction in monitoring, freeing the investor for other risk versus risk-free asset investigations and necessitating minor adjustments to trading frequency and thus changes in portfolio size. For simplicity, while numerous rules may be proposed based on changes to market risk and return realized, this application considers only the following three rules from the perspective of a risk-averse trader:

\(R_1\) : If within the upper and lower bounds and a positive change occurred in the percentage of risk with a negative change in return observed since the last trade, then decrease significantly the portfolio size.

\(R_2\) : If within the upper and lower bounds and a positive change occurred in the percentage of risk with no change in return since the last trade, then decrease slightly the portfolio size.

\(R_3\) : If within the upper and lower bounds and a negative change occurred in the percentage of risk with positive change in return observed since the last trade, then increase significantly the portfolio size.

For the risk averse investor, the assumption is made that 12% risk will be preferable with a return on investment of 20%. However, further assuming that it is desired that the portfolio risk/return be clustered within 1-sigma (.028) of the mean (.04) function \(F_1\) will be deemed “Acceptable” as indicated in Figure 3.

![Figure 2: Interval Plotting of Risk/Return for 20-Day Portfolio](image-url)
"Acceptable" According to Proportion Risk/Return

Moderately Acceptable" indicates some state of flux such that the risk/return proportion on most days are within 2-sigma of the mean. Although some may be close to the UCL or outside of the UCL this is generally only one or two stocks at high risk or low return impacting the portfolio. Thus F₂ would be the convex function shown in Figure 4.

Finally, “Unacceptable” would be any portfolio with consistent risk/return proportions above the 3-sigma UCL such that F₃ would be described by Figure 5.

With the "if" part of the rule defined as above, we define the "then" part to be the portfolio size necessary for assuring risk/return by the representation for S_j(z) as given in Figures 6 through 9.
To determine the firing strength of each rule, \( T_j \), we combine the convex functions and evaluate each rule with respect to some risk/return proportion, say 0.060 (3 poorly performing stocks in a portfolio of 50) when it was known that at the last trading day of the month the risk/return percentage was higher at 0.08 due to increased risk.

Next, for each \( R_j \) an interval on the convex set and the mean of the interval is defined such that for \( R_1 \), \( a_1 = 0 \) and \( b_1 = 35.292 \) with \( m_1 = 17.646 \); and for \( R_2 \), \( a_2 = 46.7855 \) and \( b_2 = 53.2145 \) with \( m_2 = 50 \). This yields, \( Z^* = \{ (0.1177)(17.646) + (0.3571)(50) / (0.1177 + 0.3571) \} = 41.9796 \) units which indicates that the portfolio size can be reduced to 42 investments, since the proportion of risk/return is within the limits, but barely so.

In the preceding model, the decision maker is reacting to the quantified rules determined by the fuzzy controller. Yet, the decision maker will be the one to pay the penalty if the controller selects an inappropriate rule. The penalty for most investors is obviously lost revenue and increased cost of trading to react to market risk.

If we investigate the situation where the portfolio size as suggested by the controller based on firing strength is reduced to 42 from 50 stocks, but should not have been so drastically reduced, we have the situation where \( R_1 \) was applied when \( R_2 \) or even \( R_3 \) should have been applied. Assuming that revenue generated from rapid trading is potentially much less than that of maintaining functioning stocks, the loss could be, for example, $600.00 per day or $27,000 over the 20-day period if the portfolio size is reduced (\( L_{12} = 27,000 \)). Reducing the portfolio size when it should have been increased could be higher in lost revenue than simply the cost of massive trading, for example, $1000.00 per day or $45,000 for the period (\( L_{13} = 45,000 \)). Increasing the portfolio size in order to protect from excessive risk would be the cost of increased trading. The added cost of increasing the trading could be $160 per day or $7200 for the period (\( L_{21} = 7200 \)). Maintaining the present portfolio size when it could have been reduced could be $3600 (\( L_{31} = 3600 \)). (For simplicity sake, crisp values have been used to represent the losses. The model does, of course, allow losses to be fuzzy values.)

The decision maker may believe that the likelihood of reducing the portfolio size inappropriately instead of maintaining the 50 units or increasing the portfolio size as a precautionary measure is: \( p_{12} = 0.2/18,900 + 0.3/14,400 \), \( p_{13} = 0.9/0 + 0.8/1 \), respectively. This indicates that the decision maker believes that there is a significant likelihood of a penalty being incurred if \( R_1 \) is applied inappropriately and the portfolio size is reduced, but has little belief that reducing the portfolio size is the wrong decision and, thus a low belief that the loss will be incurred. Similarly, the second fuzzy probability indicates that the decision maker believes strongly that no penalty will be incurred due to a wrong decision to reduce the portfolio size (\( R_3 \)) rather than increase the portfolio size to at least 60 (\( R_3 \)). All of this indicates that, in general, the decision maker does not think the portfolio size needs to remain at 50 units and very strongly that the portfolio size does not need to be increased.

It should be noted that \( p_0 \) is not necessarily the same as \( p_i \). The uncertainty of the decision maker with respect to selecting \( R_1 \) when \( R_i \) should be applied is not the same as applying \( R_1 \) when \( R_i \) should be applied. Similarly, the penalty of selecting \( R_i \) instead of \( R_1 \) is not the same as the penalty of \( R_i \) instead of \( R_1 \). Again, differences in costs and uncertainty relate to the phase in the portfolio management process.

According to the scenario above:

\[
\begin{align*}
E(L_1) &= L_{12}p_{12} + L_{13}p_{13} \\
&= (27,000)(0.2/18,900 + 0.3/14,400) + (45,000)(0.9/0 + 0.8/1) \\
&= 2/18,900 + 0.3/14,400 + 9/0 + 0.8/4500 \\
E(L_2) &= L_{21}p_{21} = 36,000(0.6/3 + 0.5/4) = 6.1080 + 0.5/1440 \\
E(L_3) &= L_{31}p_{31} = (7200)(0.1/0) \\
M &= \{ 0, 1080, 1440, 16200, 18900 \}
\end{align*}
\]
This indicates that the potential for loss has had an impact on the portfolio size such that the decision maker would reduce the portfolio size, but not as dramatically as originally indicated by the fuzzy controller. The loss-based fuzzy controller recommended in this paper reacts to the low belief that reducing the sample size is the wrong decision, while also considering that if the portfolio size is reduced to 42 stocks, the probability of incurring a loss is relatively high.

In contrast, if \( p_{12} = .7/2 + .8/3 \) such that the decision maker believes that \( R_1 \) may be the wrong decision but a relatively low penalty will be incurred, \( Z^* = 47.60 \). Thus, the portfolio size would be only moderately reduced, and \( R_2 \) would take precedence over \( R_1 \). For this model, \( R_1 \) has no impact on the portfolio size since \( T_1 = 0 \).

Note that if the decision maker believes that the appropriate rule has been applied by the controller, \( p_j = 0 \) for all \( i \neq j \) then \( E(L_i) = 0 \) for \( i = 1,2,3 \). Then \( a_1 = 0 \) and \( b_2 = 1 \) such that \( T_j = b_1T_j \) indicates that \( T_j = 0 \), such that \( Z^* = Z^* \). For the example, the rule would be to reduce the portfolio size according to both the fuzzy controller and the loss-based fuzzy controller.

V. CONCLUSIONS

The application as presented is obviously incomplete. Other rules may be used that relate to the fact that most of the portfolio stocks are within bounds found acceptable to the decision maker. If most of the stocks are yielding satisfactory risk/return ratios, the decision maker may decide to hold these, incurring more risk on some that he/she believes may return higher revenue at a later time. The cost of trading and frequency thereof also impacts the decision to allow the fuzzy controller to adjust automatically on a daily basis. The rules and frequency of firing are at the discrimination of the decision maker. The fuzzy controller may, therefore, be set to control and adjust or as simply a monitoring technique. The addition of the concept of the fuzzy loss-based controller puts more decision power into the hands of the investor. However, in order to be a strength of the model, the values determined (unlike simply setting them as in the example) must be realistic. This will require considerable investigation by the decision maker, perhaps utilizing simulations, linear programming or other stochastically based models or other fuzzy logic models to address the uncertainty of the loss figures.

The model proposed is capable of handling any number of rules, and any method for determination of loss. It can be tailored easily to the investor and reacts to the risk-taking or risk-averseness of the person managing his/her personal portfolio.

REFERENCES


