ψ-Eventual Stability of Differential System with Impulses

Bhanu Gupta

Abstract—In this paper, the criteria of ψ-eventual stability have been established for generalized impulsive differential systems of multiple dependent variables. The sufficient conditions have been obtained using piecewise continuous Lyapunov function. An example is given to support our theoretical result.

Keywords—Impulsive differential equations, Lyapunov function, Eventual stability.

I. INTRODUCTION

Many evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. The impulsive system of differential equation is an adequate apparatus for the mathematical simulation of numerous processes and phenomena studied in biology, economics and technology etc. That is why, in recent years, the study of such systems has been very intensive (See [2-11]).

Akinbile [7] introduced the notion of ψ-stability of degree k with respect to a function ψ ∈ C[R1, R+], increasing and differentiable on R+, where R+ = [0, ∞) and such that ψ(t) ≥ 1 for t ≥ 0 and limt→∞ ψ(t) = b, b ∈ [1, ∞). In [6], Morchalo introduced the notions of ψ-stability, ψ-uniform stability and ψ-asymptotic stability of trivial solution of the nonlinear system x' = f(t, x). Then Dianadescu [1] proved some sufficient conditions for ψ-stability of the zero solution of a non-linear Volterra integro-differential system.

The main purpose of this work is to investigate the sufficient conditions for the existence of ψ-eventual stability of trivial solution for generalized impulsive differential system of multiple dependent variables, where ψ is a matrix function defined in the section below.

The paper is organized as follows. In Section 2, we introduce some preliminary definitions and notations which will be used throughout the paper. In Section 3, we investigate some sufficient conditions for ψ-uniform eventual stability and ψ-uniform asymptotic eventual stability of trivial solution of the impulsive differential systems. In Section 4, an example to support our theoretical result has been discussed.

II. PRIIMARIES

Let R^n denote the Euclidean n-space. Elements of this space are denoted by x = (x_1, x_2, ..., x_n)^T and their norm is given by ||x|| = max{|x_1|, |x_2|, ..., |x_n|}. For n x n real matrices, we define the norm |A| = sup_{||x||≤1} ||Ax||. Let

Ψ : R_+ → (0, ∞), i = 1, 2, ..., n where R_+ = [0, ∞) be continuous functions and let Ψ = diag{Ψ_1, Ψ_2, ..., Ψ_n}.

Let R^n be the s-dimensional Euclidean space with a suitable norm ||·|| and R^n = {x ∈ R^n : ||x|| < H}.

Consider the system

\[ \begin{align*}
\dot{x} &= f(t, x) + g(t, y) + h(t, z), t \neq \tau_k, \\
\dot{y} &= u(t, x, y, z) + v(t, y, z) + w(t, x, z), t \neq \tau_k, \\
\dot{z} &= l(t, x, y, z), t \neq \tau_k, \\
\Delta x|_{t=\tau_k} &= A_k(x) + B_k(y) + C_k(z), \\
\Delta y|_{t=\tau_k} &= D_k(x, y) + E_k(y, z) + F_k(z, x), \\
\Delta z|_{t=\tau_k} &= G_k(x, y, z), k = 1, 2, \ldots,
\end{align*} \]

where t ∈ R_+, x ∈ R^n, y ∈ R^n, z ∈ R^n, f : R_+ × R^n × R^n → R^n, g : R_+ × R^n → R^n, h : R_+ × R^n × R^n → R^n, u : R_+ × R^n × R^n × R^n × R^n → R^n, v : R_+ × R^n × R^n × R^n → R^n, w : R_+ × R^n × R^n × R^n → R^n, l : R_+ × R^n × R^n × R^n → R^n, A_k : R^n → R^n, B_k : R^n → R^n, C_k : R^n → R^n, D_k : R^n × R^n → R^n, E_k : R^n × R^n → R^n, F_k : R^n × R^n → R^n, G_k : R^n × R^n × R^n → R^n.

\[ \begin{align*}
\Delta x|_{t=\tau_k} &= x(\tau_k) - x(\tau_k^-), \\
\Delta y|_{t=\tau_k} &= y(\tau_k) - y(\tau_k^-), \\
\Delta z|_{t=\tau_k} &= z(\tau_k) - z(\tau_k^-).
\end{align*} \]

Let τ_0 ∈ R_+, x_0 ∈ R^n, y_0 ∈ R^n, z_0 ∈ R^n.

Let x(t_0, x_0, y_0, z_0), y(t_0, t_0, x_0, y_0, z_0), z(t_0, t_0, x_0, y_0, z_0) be the solution of the system (1) satisfying the initial conditions

\[ \begin{align*}
x(t_0^+) &= x_0, \\
y(t_0^+) &= y_0, \\
z(t_0^+) &= z_0.
\end{align*} \]

Throughout this article, we assume the following conditions:

(a) The functions f(t, x), g(t, y), h(t, z), v(t, x, y), v(t, y, z), w(t, x, z) and l(t, x, y, z) are continuous in their definition domains, f(t, 0) = 0 = g(t, 0) = h(t, 0) = 0; u(t, 0, 0) = v(t, 0, 0) = w(t, 0, 0) = 0; l(t, 0, 0, 0) = 0 for t_0 ∈ R_+.

(b) The functions A_k, B_k, C_k, D_k, E_k, F_k and G_k are continuous in their definition domains and A(0) = B(0) = C(0) = D(0) = E(0) = F(0) = G(0) = 0.

(c) If x ∈ R^n, y ∈ R^n and z ∈ R^n, then

\[ \begin{align*}
||x + A_k(x) + B_k(y) + C_k(z)|| &\leq ||x||, \\
||y + D_k(x, y) + E_k(x, y) + F_k(z, y)|| &\leq ||y||, \\
||z + G_k(x, y, z)|| &\leq ||z||.
\end{align*} \]

(d) 0 ≤ τ_0 < τ_1 < τ_2 < ... and lim_{k→∞} τ_k = ∞.

(e) For each point (t_0, x_0, y_0, z_0) ∈ R_+ × R^n × R^n × R^n, the solution x(t, t_0, x_0, y_0, z_0), y(t, t_0, x_0, y_0, z_0), z(t, t_0, x_0, y_0, z_0) of the system (1) is defined in (t_0, ∞) and is unique.

Note that Ψ(t_0) = Ψ_0.

Now, we have following definitions:
Definition 2.1: The zero solution of (1) is said to be $\Psi$-uniformly eventually stable if for $\epsilon > 0$, $\delta = \delta(\epsilon) > 0$ and $\tau = \tau(\epsilon) > 0$ such that $\|\Psi(t)x(t)+\Psi(t)y(t)+\Psi(t)z(t)\| < \epsilon$ for $\|\Psi(0)x_0+\Psi(0)y_0+\Psi(0)z_0\| < \delta$ and $t \geq t_0 \geq \tau(\epsilon)$.

Definition 2.2: The zero solution of (1) is said to be $\Psi$-uniformly asymptotically eventually stable if it is uniformly eventually stable and $\exists \delta > 0$ such that for $\epsilon > 0$, there exist $\tau = \tau(\epsilon) > 0$ and $T = T(\epsilon) > 0$ such that for $(x_0, y_0, z_0) \in R_{R} \times R_{R} \times R_{R}$ and $\|\Psi(0)x_0+\Psi(0)y_0+\Psi(0)z_0\| < \delta$ implies $\|\Psi(t)x(t)+\Psi(t)y(t)+\Psi(t)z(t)\| < \epsilon$ for $t \geq t_0 + T, t_0 \geq \tau(\epsilon)$.

Definition 2.3: A function $V : R_+ \times R_{R} \times R_{R} \times R_{R} \rightarrow R_+$ is said to belong to class $\mathcal{V}_0$ if

(i) $V$ is continuous on each of the sets $[\tau_{k-1}, \tau_k) \times R_{R} \times R_{R} \times R_{R}$;

(ii) $V(t, x, y, z)$ is locally Lipschitzian in all $x, y, z$ on each of the sets $[\tau_{k-1}, \tau_k) \times R_{R} \times R_{R} \times R_{R}$ and $V(t, 0, 0, 0) = 0$ for $t \in R_+$;

(iii) For each $(x, y, z) \in R_{R} \times R_{R} \times R_{R}$, we have, $\lim_{(x', y', z') \rightarrow (x, y, z)} V(t, x', y', z') = V(t, x, y, z) \in V_0(\tau_{k-1}, \tau_k, 0)$ exists.

Definition 2.4: Let $V \in \mathcal{V}_0$, for any $(t, x, y, z) \in [\tau_{k-1}, \tau_k) \times R_{R} \times R_{R} \times R_{R}$, the right hand derivative $V'(t, x, y, z)$ along the solution of the problem (1) is defined as

$V'(t, x, y, z, z(t)) = \lim_{s \rightarrow 0^+} \frac{1}{s} \left[ V(t + s, x + s f(t, x) + g(t, y) + h(t, z), y + s w(t, x, y) + v(t, y, z) + w(t, x, z), z + s l(t, x, y, z) - V(t, x, y, z) \right].$

We define, $K = \{ w \in C(R_+, R_+) : w$ is strictly increasing and $w(0) = 0 \}$, $K_1 = \{ \phi \in C(R_+, R_+) : \phi$ is increasing and $\phi(s) < s$ for $s > 0 \}$.

III. MAIN RESULTS

In this section, we shall present sufficient conditions for the $\Psi$-uniform eventual stability and $\Psi$-uniform asymptotic eventual stability of trivial solution of the impulsive differential system (1).

Theorem 3.1: Assume that there exist functions $V \in \mathcal{V}_0$, $a, b \in K, \phi \in K_1$ such that

(i) $b(\|\Psi(t)x(t)+\Psi(t)y(t)+\Psi(t)z(t)\|) \leq V(t, x, y, z) \leq a(\|\Psi(t)x(t)+\Psi(t)y(t)+\Psi(t)z(t)\|),$

(ii) $V(t, x, y, z, z(t)) \leq g(t)w(V(t, x, y, z, z(t))),$

(iii) $V(t, x, y, z) \leq g(t)w(V(t, x, y, z, z(t))),$

(iv) For all $k \in N, (a, b, c) \in R_{R} \times R_{R} \times R_{R}$, $V(t, x, y, z) \leq a(t)w(V(t, x, y, z, z(t))),$

and the functions $g, w : R_+ \rightarrow R_+$ are locally integrable,

From condition (iii), we have

$V(t, x, y, z) \leq g(t)w(V(t, x, y, z, z(t))),$

and

$V(t, x, y, z) \leq g(t)w(V(t, x, y, z, z(t))),$

for all $k \in N, (a, b, c) \in R_{R} \times R_{R} \times R_{R}$, $V(t, x, y, z) \leq a(t)w(V(t, x, y, z, z(t))),$

From condition (iii), we have

$V(t, x, y, z) \leq g(t)w(V(t, x, y, z, z(t))),$

and

$V(t, x, y, z) \leq g(t)w(V(t, x, y, z, z(t))),$

for all $k \in N, (a, b, c) \in R_{R} \times R_{R} \times R_{R}$, $V(t, x, y, z) \leq a(t)w(V(t, x, y, z, z(t))),$

and

$V(t, x, y, z) \leq g(t)w(V(t, x, y, z, z(t))),$

for all $k \in N, (a, b, c) \in R_{R} \times R_{R} \times R_{R}$, $V(t, x, y, z) \leq a(t)w(V(t, x, y, z, z(t))),$

and

$V(t, x, y, z) \leq g(t)w(V(t, x, y, z, z(t))),$
From the continuity of $V(t, x, y, z)$ in $[\tau_m, \tau_{m+1})$, there is an $r_1 \in (\tau_m, \ell)$ such that
\[ V(r_1, x(r_1), y(r_1), z(r_1)) = \phi^{-1}(a(\delta)), \text{ for } t \leq \ell. \]
\[ V(t, x(t), y(t), z(t)) \leq \phi^{-1}(a(\delta)), t_0 \leq t \leq r_1, \]
and there also exist an $r_2 \in (\tau_m, r_1)$ such that
\[ V(r_2, x(r_2), y(r_2), z(r_2)) = a(\delta), \]
\[ V(t, x(t), y(t), z(t)) \geq a(\delta), r_2 \leq t \leq r_1. \]

Again integrating the inequality given in (ii) within $[r_2, r_1]$ and by similarly as above, we get a contradiction.

So the inequality (9) holds.

From (iii), we have
\[ V(\tau_{m+1}, x(\tau_{m+1}), y(\tau_{m+1}), z(\tau_{m+1})) = V(\tau_m, x(\tau_m), y(\tau_m), z(\tau_m)) + \int_{\tau_m}^{\tau_{m+1}} A(t) x(t) + B(t) y(t) + C(t) z(t) dt \]
\[ + \int_{\tau_m}^{\tau_{m+1}} \left( D_1(y, z) + D_2(y, z) + D_3(y, z) \right) dt \leq \phi(V(\tau_m, x(\tau_m), y(\tau_m), z(\tau_m)) + 1) \]
\[ \leq \phi(\phi^{-1}(a(\delta))) = a(\delta). \]

By induction, we can prove that in general
\[ V(t, x(t), y(t), z(t)) < \phi^{-1}(a(\delta)), \tau_m < t < \tau_{m+1}, \]
\[ V(\tau_{m+1}, x(\tau_{m+1}), y(\tau_{m+1}), z(\tau_{m+1})) \leq a(\delta), \]
for $i = 0, 1, 2, \ldots$

As $a(\delta) < \phi^{-1}(a(\delta))$, it follows that form (3) and (13) that
\[ V(t, x(t), y(t), z(t)) < \phi^{-1}(a(\delta)) < a(\delta), t > t_0 \geq \tau(\epsilon). \]

Thus the zero solution of (1) is $\Psi$-uniformly eventually stable.

**Theorem 3.2** Let all the conditions of Theorem 3.1 be satisfied except (iv), which is replaced by
\[ (v) \ r = \sup_{x \in E_{(\tau_k, \tau_k-\epsilon)}} \|x(t) - x(t)\| < \infty, \ A = \sup_{t \geq 0} \|f(t) \| \leq \infty \text{ and } B = \inf_{t > 0} \|f(t) \| \leq \infty. \]

Then the zero solution of system (1) is $\Psi$-uniformly asymptotically eventually stable.

**Proof:** If all the conditions of Theorem 3.2 holds, then all the conditions of Theorem 3.1 hold. Thus the zero solutions of system (1) is $\Psi$-uniformly stable.

Therefore, for given $q > 0$, for all $t_0 \in R_+$, we can choose $\delta > 0, \tau(q) > 0$ such that $a(\delta) = \phi(b(\delta))$, for all $(x_0, y_0, z_0) \in R_+^3 \times R_+^3 \times R_+^3$, such that $\|x_0, y_0, z_0\| < \delta$ implies
\[ \|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\| \leq q, t \geq t_0 \geq \tau(q). \]

Moreover, $V(t, x, y, z) \leq a(\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)) \leq a(q), t \geq t_0 \geq \tau(q).

Now, let $\epsilon > 0$ be given, we can find $\tau(\epsilon) > 0$ such that $t_0 \geq \tau(\epsilon).

If $\tau(\epsilon) \leq \tau(q)$, then $V(t, x, y, z) \leq a(\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)) \leq a(q), t \geq t_0 \geq \tau(q) \geq \tau(\epsilon).

If $\tau(\epsilon) > \tau(q)$, then as $V(t, x, y, z) \leq a(\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)) \leq a(q), t \geq t_0 \geq \tau(q), \Psi(t)y(t) + \Psi(t)z(t)) \leq a(q), t \geq t_0 \geq \tau(q),$ it is obvious that $V(t, x, y, z) \leq a(\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)) \leq a(q), t \geq t_0 \geq \tau(q).

So in any case, we have $V(t, x, y, z) \leq a(\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)) \leq a(q)$ holds for $t \geq t_0 \geq \tau(q).

In the following, we prove that for $T(\epsilon) > 0$ such that $\|x_0 + y_0 + z_0\| < \delta$ implies $\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\| \leq \epsilon$, for $\epsilon > 0$.

Now, let $M = M(\epsilon) = \sup \frac{1}{\|x(t)\|} \leq s \leq a(\delta) \leq a(q)$ and note that $0 < M < \infty$. For $b(\epsilon) \leq p \leq a(\epsilon)$, we have $\phi(b(\epsilon)) \leq \phi(p) \leq a(\delta)$ and so $B \leq a(\delta) = \sup_{t \geq 0} \|x(t)\| \leq a(\epsilon)$. From which we obtain $\phi(b(\epsilon)) \leq p - B/M \leq a(\epsilon)$, where $d = d(\epsilon) > 0$ is chosen such that $d < B/4$.

Let $N = N(\epsilon)$ be the smallest positive integer for which $a(\epsilon) < b(\epsilon) + Nd$ and we define $T(\epsilon) = N^2$. Given a solution $x(t) = x(t, t_0, x_0, y_0, z_0)$ of system (1), where $t_0 \in [\tau_1, \tau_2]$, for some integer $i$, we will prove that $\|x_0 + y_0 + z_0\| \leq \delta$ then $\|\Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t)\| \leq a(\delta)$, for $t \geq t_0 \geq \tau(q).

Clearly, if $0 < \delta < \infty$ and $\delta \geq 1$, we will show that

(a) if $V(t, x(t), y(t), z(t)) \leq M$ then $V(t, y(t), z(t)) \leq M$ for $t \geq \tau(q)$,

(b) if in addition $M < b(\epsilon)$, then $V(t, x(t), y(t), z(t)) \leq D - d$ for $t \geq \tau(q).

Firstly we prove (a).

If (a) does not holds, then there exist some $t \geq \tau(q)$ such that $V(t, x(t), y(t), z(t)) > D$. Then let $t_1 = \inf \{t \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q) \}$, for some $k \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q).

As $V(t, x(t), y(t), z(t)) \leq a(\epsilon) \leq a(\epsilon)$, we have $\phi(b(\epsilon)) \leq p - B/M \leq a(\epsilon)$, where $d = d(\epsilon) > 0$ is chosen such that $d < B/4$.

Let $\overline{T} = \sup \{t \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q) \geq \tau(q).

As $V(t, x(t), y(t), z(t)) \leq a(\epsilon) \leq a(\epsilon)$, we have $\phi(b(\epsilon)) \leq p - B/M \leq a(\epsilon)$, where $d = d(\epsilon) > 0$ is chosen such that $d < B/4$.
and let \( k \geq j \) be chosen such that \( \tau_k \in [\tau_j, \tau_{j+1}) \). As 
\[
b(t) \leq D \leq a(q), \text{ we have } \phi(D) < D - d.
\]
So from (a) and condition (iii), 
\[
V(\tau_k, x(\tau_k), y(\tau_k), z(\tau_k)) \leq \phi(V(\tau_k, x(\tau_k), y(\tau_k), z(\tau_k))) 
\]
\[
\leq \phi(D) < D - d.
\]
Thus \( \tau_k \in [\tau_j, \tau_{j+1}). \)
Moreover, \( V(\tau_j, x(\tau_j), y(\tau_j), z(\tau_j)) = D - d \) and for 
\[
t \in [\tau_j, \tau_{j+1}), V(t, x(t), y(t), z(t)) \leq \phi(D).
\]
Let \( \mathcal{F} = \sup \{ t \in [\tau_j, \tau_{j+1}) \mid V(t, x(t), y(t), z(t)) \leq \phi(D) \} \) and 
\[
V(t_j, x(t), y(t), z(t)) \geq \phi(D) \quad \text{for } t \in [\tau_j, \tau_{j+1}) \text{ and all } x(t), y(t), z(t).
\]
So if we integrate the inequality \( V(t, x(t), y(t), z(t)) \leq g(t)w(V(t, x(t), y(t), z(t))) \) over \([\tau_j, \tau_{j+1})\), we have 
\[
\int_{\tau_j}^{\tau_{j+1}} V(t, x(t), y(t), z(t)) \frac{ds}{w(s)} \leq A.
\]
Also, 
\[
\int_{\tau_j}^{\tau_{j+1}} V(t, x(t), y(t), z(t)) \frac{ds}{w(s)} = \int_{\tau_j}^{\tau_{j+1}} \frac{d}{dt} \frac{\phi(D)}{w(t)} dt
\]
\[
= \phi(D) - \phi(D) \frac{D - d}{w(t)}.
\]
As \( b(e) \leq D \leq a(q) \), we have 
\[
\phi(b(e)) \leq \phi(D) < D - d = \phi(D) \leq a(q).
\]
Thus, \( \frac{1}{w(t)} \leq M \) for \( D - d \leq s \leq D \).
So we get 
\[
\int_{\tau_j}^{\tau_{j+1}} V(t, x(t), y(t), z(t)) \frac{ds}{w(s)} \geq B - \int_{\tau_j}^{\tau_{j+1}} M ds = B - dM
\]
\[
> B + A - B = A,
\]
which is a contradiction, so (b) holds.
We define the indices \( k(i) \) for \( i = 1, 2, 3, \ldots, N \) as follows.
Let \( k(1) = l \) and for \( i = 2, \ldots, N, \) let \( k(i) \) be chosen such that \( \tau_{k(i-1)} < \tau_{k(i)} < \tau_{k(i+1)} \).
Then from condition (i), we have \( \tau_{k(i)} = t_0 + r, \) and for 
\[
i = 1, 2, \ldots, N,
\]
\[
\tau_{k(i)} \leq t_0 + r + \tau_{k(i-1)} + r.
\]
Combining these inequalities, 
\[
\tau_{k(i)} \leq t_0 + r + \tau_{k(i-1)} + r.
\]
We claim that for all \( i = 1, 2, \ldots, N, V(t, x(t), y(t), z(t)) \leq a(q) - i d \) for \( t \geq \tau_{k(i)} \).
By setting \( D = a(q) \) in (b), by condition (iii) and \( b(e) \leq a(q), \) 
we get \( V(t, x(t), y(t), z(t)) \leq a(q) - d \) for \( t \geq \tau_{k(i)} \) as 
\[
V(t, x(t), y(t), z(t)) \leq a(q) \quad \text{for } t \in [t_0, \tau_{k(i)}),
\]
which establish the base case.
We now proceed by induction and assume that 
\[
V(t, x(t), y(t), z(t)) \leq a(q) - j d \quad \text{for } t \geq \tau_{k(j)}
\]
for some \( 1 \leq j \leq N - 1. \)
Let \( D = a(q) - j d \) as \( \tau_{k(j)} - \tau_{k(j-1)} \leq D \) for \( t \geq \tau_{k(j)} \) and so 
\[
V(t, x(t), y(t), z(t)) \leq D - d = a(q) - (j + 1) d \quad \text{for } t \geq \tau_{k(j+1)}.
\]
So we have proved our claim by induction.
When \( j = N - 1, \) we get 
\[
V(t, x(t), y(t), z(t)) \leq a(q) - Nd < b(e), \quad t \geq \tau_{k(N)}.
\]
As \( t_0 + T(e) \geq \tau_{k(N)} \), by condition (ii), we get \( \| \Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t) \| < \epsilon \) for \( t \geq t_0 + T(e) \) and \( t_0 \geq \tau(e). \)
Thus, the impulsive system (1) is \( \Psi \)-uniformly asymptotically eventually stable.

**IV. Example**

In this section, we give an example to illustrate our theoretical results.
Consider the system
\[
\begin{align*}
\dot{x} &= cx(t) + dy(t) + ez(t), \quad t \neq \tau_k, \\
&= c_1y(t) + d_1z(t), \quad t \neq \tau_k, \quad y(\tau_k) = c_1y(\tau_k) + \beta_1z(\tau_k), \\
\dot{y} &= c_1y(t) + d_1z(t), \quad t \neq \tau_k, \quad y(\tau_k) = c_1y(\tau_k) + \beta_1z(\tau_k), \\
\dot{z} &= c_1z(t), \quad t \neq \tau_k, \quad z(\tau_k) = \gamma_1z(\tau_k), \quad k = 1, 2, 3, \ldots.
\end{align*}
\]
where \( 0 \leq \tau_0 < \tau_1 < \tau_2 \leq \tau_3 \ldots \) and \( k \to \infty, \quad \tau_k = \infty, \quad c > 0, \quad d > a_2 \), \( e > 0, \quad c_1 > 0, \quad d_1 > 0, \quad c_1 > 0, \quad a > 0, \quad \beta > 0, \quad \gamma > 0, \quad \alpha_1 > 0, \quad \beta_1 = 0, \quad \gamma_1 = 0, \) and the following conditions hold:
(1) \( c > c_1, \quad d > d_1, \quad d > e, \quad a^2 > \beta^2 + \alpha_1^2, \quad \alpha^2 > \beta^2 + \gamma^2 = \gamma^2 \)
(2) \( \tau_k - \tau_{k-1} < \frac{\ln(a_2 + b_2)}{a_2 + b_2} \)
Let \( V(t, x, y, z) = \frac{\gamma^2 x^2 + y^2 + z^2}{2} \), \( \phi(s) = (2a^2 + \beta^2)s, \quad w(s) = s, \quad g(t) = 2a + d. \)
Take \( \Psi(t) = 1/2, \quad a(x) = 4x^2, \quad b(x) = x/2 \in K. \)
Clearly, 
\[
\begin{align*}
\mathcal{B}(\| \Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t) \|) &= \left( \frac{x^2 + y^2 + z^2}{2} \right), \\
\mathcal{B}(\| \Psi(t)x(t) + \Psi(t)y(t) + \Psi(t)z(t) \|) &= \left( \frac{x^2 + y^2 + z^2}{2} \right)
\end{align*}
\]
Now,
\[
\begin{align*}
V'(t, x, y, z) &= \dot{x}x + \dot{y}y + \dot{z}z \\
&= cx^2(t) + dy^2(t) + ez^2(t), \\
&= c_1x^2(t) + d_1y^2(t) + c_1y^2(t) + d_1z^2(t) + c_1y^2(t) + d_1z^2(t) \\
&\leq c_1x^2(t) + d_1y^2(t) + c_1y^2(t) + d_1z^2(t) \frac{c_1^2 + d_1^2 + c_1^2 + d_1^2}{2} \\
&= g(t)w(V(t, x, y, z)).
\end{align*}
\]
Also,

\[ V(\tau_k, x(\tau_k^-)) + A_t(x) + B_t(y) + C_t(z), y(\tau_k^-) + D_t(x, y) \\
+ E_t(y, z) + F_t(x, z), z(\tau_k^-) + G_t(x, y, z) \\
= V(\tau_k, \alpha x(\tau_k^-) + \beta y(\tau_k^-) + \gamma z(\tau_k^-), \alpha \beta x(\tau_k^-) \\
+ \beta \gamma y(\tau_k^-) + \gamma \alpha z(\tau_k^-)) \\
= \frac{1}{2} [\alpha^2 x^2(\tau_k^-) + \beta^2 y^2(\tau_k^-) + \gamma^2 z^2(\tau_k^-) + 2\alpha \beta x(\tau_k^-)y(\tau_k^-) \\
+ 2\beta \gamma y(\tau_k^-)z(\tau_k^-) + 2\alpha \gamma x(\tau_k^-)z(\tau_k^-) + \alpha^2 y^2(\tau_k^-) \\
+ \beta^2 z^2(\tau_k^-) + 2\alpha \beta \gamma x(\tau_k^-)z(\tau_k^-) + \gamma^2 z^2] \\
\leq \frac{1}{2} [\alpha^2 x^2(\tau_k^-) + y^2(\tau_k^-) + z^2(\tau_k^-)] + 2\alpha \beta [x^2(\tau_k^-) \\
+ y^2(\tau_k^-) + z^2(\tau_k^-)] \\
= \frac{\alpha^2 + 2\alpha \beta}{2} [x^2(\tau_k^-) + y^2(\tau_k^-) + z^2(\tau_k^-)] \\
\leq (2\alpha^2 + \beta^2)[x^2(\tau_k^-) + y^2(\tau_k^-) + z^2(\tau_k^-)] \\
= \phi (V(\tau_k, x(\tau_k^-), y(\tau_k^-), z(\tau_k^-))).
\]

Now, let \( A = -\ln \frac{2\alpha^2 + \beta^2}{2} \), then \( A > 0 \) and

\[
\int_{\tau_n}^{\tau_{n+1}} g(s) \, ds < \frac{(c+d) - \ln(2\alpha^2 + \beta^2) + \ln 2}{2\alpha^2 + \beta^2} = -\ln \frac{2\alpha^2 + \beta^2}{2} = A.
\]

Lastly for any \( \mu > 0 \),

\[
\int_{\mu}^{\infty} \frac{1}{x(\tau_k^-)} \, dx = \int_{\mu}^{\infty} \left( \frac{1}{x(\tau_k^-)} \right)^{\frac{\mu}{2}} \, dx = \ln \frac{2\alpha^2 + \beta^2}{2} = A.
\]

Therefore by Theorem 3.1, the zero solution of system (15) is \( \Psi \)-uniformly eventually stable.

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**REFERENCES**


